GRAPHICAL NOTATION SCHEMES

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4 April 2012

WHAT'S THE ISSUE?

- Symbolic expressions used in foundational mathematics
  - Powerful methods
  - Objects of study in themselves
  - Can be technical and syntax-heavy
  - Can be easy to make mistakes by hand and hard to spot structure

WHAT'S THE ISSUE?

- Researchers have always found ways round this:
  - Doodles in margins to help symbolic calculations
  - Proofs-by-picture

WHAT'S THE ISSUE?

- Pictures can capture important aspects of abstract structure
  - It's not a coincidence that they're so useful
  - Many classes of graphs exhibit rich categorical structure
    - That's *why* they're useful!

STRING DIAGRAMS

- Just one type of example: monoidal categories with other structure
  - Nice examples to demonstrate the ideas

Fantastic survey: Peter Selinger's *A survey of graphical languages for monoidal categories*. New Structures for Physics 2011
Preprint: mathstat.dal.ca/~selinger/papers/graphical.pdf

MONOIDAL CATEGORIES

- A **monoidal category** is a category with
  - a bifunctorial **tensor product**, $\otimes$
  - A specified "tensor unit", I
  - Associativity and identity natural isomorphisms, $a, l, r,$ or strictness
  - Coherence axioms

MONOIDAL CATEGORIES

- Examples
  - $(\mathrm{Set}, \times, \{*\})$ non-strict
  - $(\mathcal{C}, \times, t)$ category with binary products and terminal object
  - Rel category of sets and relations
  - $([\mathcal{C}, \mathcal{C}], \circ, \mathrm{id}_\mathcal{C})$ strict

MONOIDAL EXPRESSIONS

Expressions in categories $\qquad A \qquad B \qquad C$

$$f : A \to B \qquad g : B \to C \qquad g \circ f : A \to C$$

$$\text{“}C\text{”} = \mathrm{id}_C : C \to C$$

Expressions in monoidal categories

$$A \otimes B \qquad C \otimes D \otimes E \otimes F$$

$$f \otimes g : A \otimes C \to B \otimes C \qquad f \otimes C : A \otimes C \to B \otimes C$$

$$h : A \otimes B \to C \otimes D \otimes E \otimes F$$

$$f : X \to X_1 \otimes \cdots \otimes X_n$$

$$g : X_1 \otimes \cdots \otimes X_n \to Y$$

$$g \circ f : X \to Y$$

MONOIDAL EXPRESSIONS

$$f : A \to E \otimes D$$

$$h : D \to G \otimes H$$

$$g : D \otimes B \otimes C \to F$$

- When are these equal *due to monoidal axioms*?
- When are these equal *in any monoidal category*?
  - (Again, working strictly)

$$(E \otimes g \otimes h) \circ (f \otimes B \otimes C \otimes D)$$

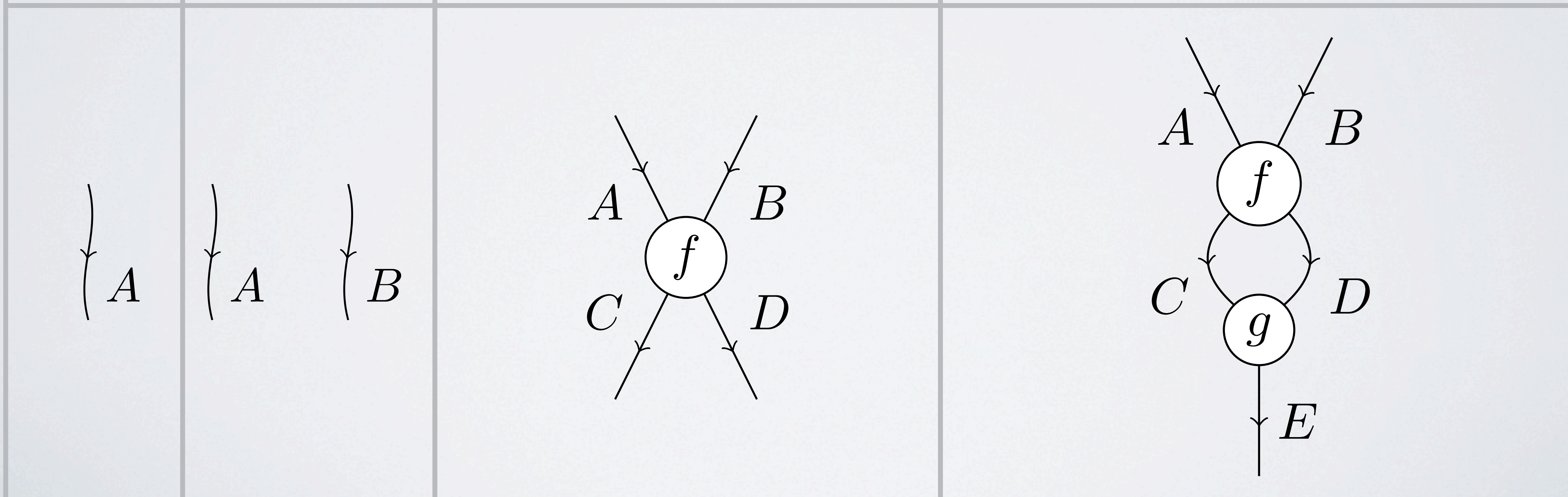
$$\| \, ?$$

$$(E \otimes g \otimes G \otimes H) \circ (f \otimes B \otimes C \otimes h)$$

STRING DIAGRAMS FOR MONOIDAL CATEGORIES

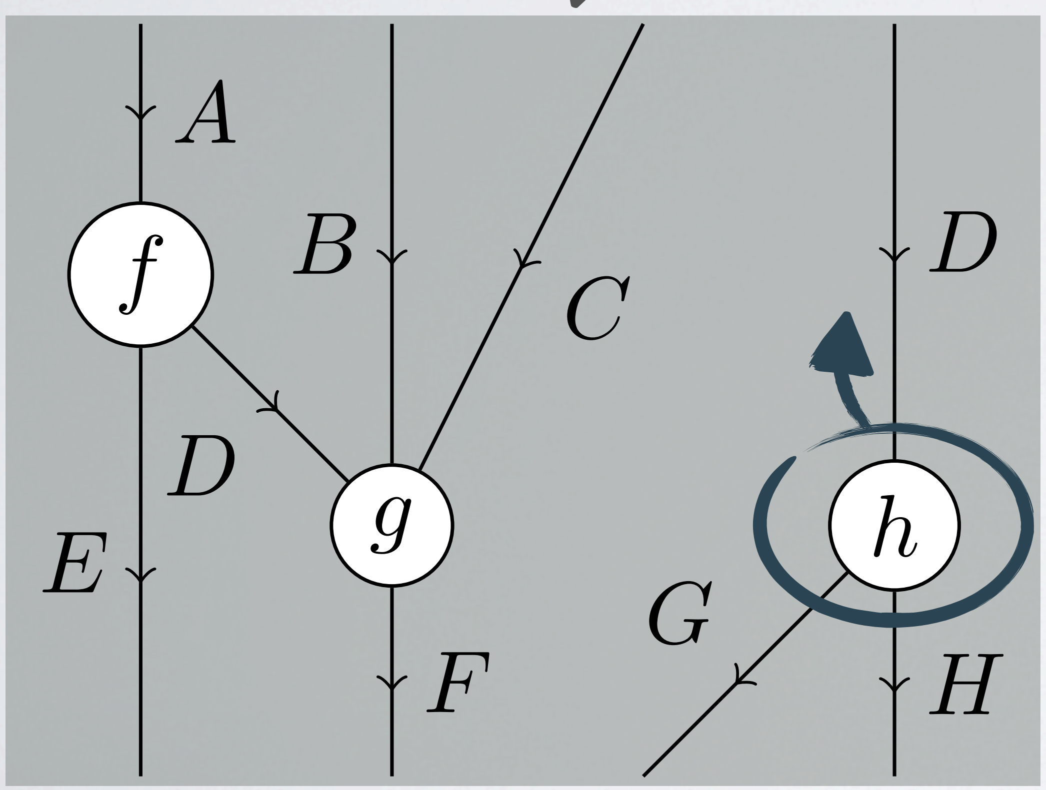
- Diagrams to represent expressions in monoidal categories

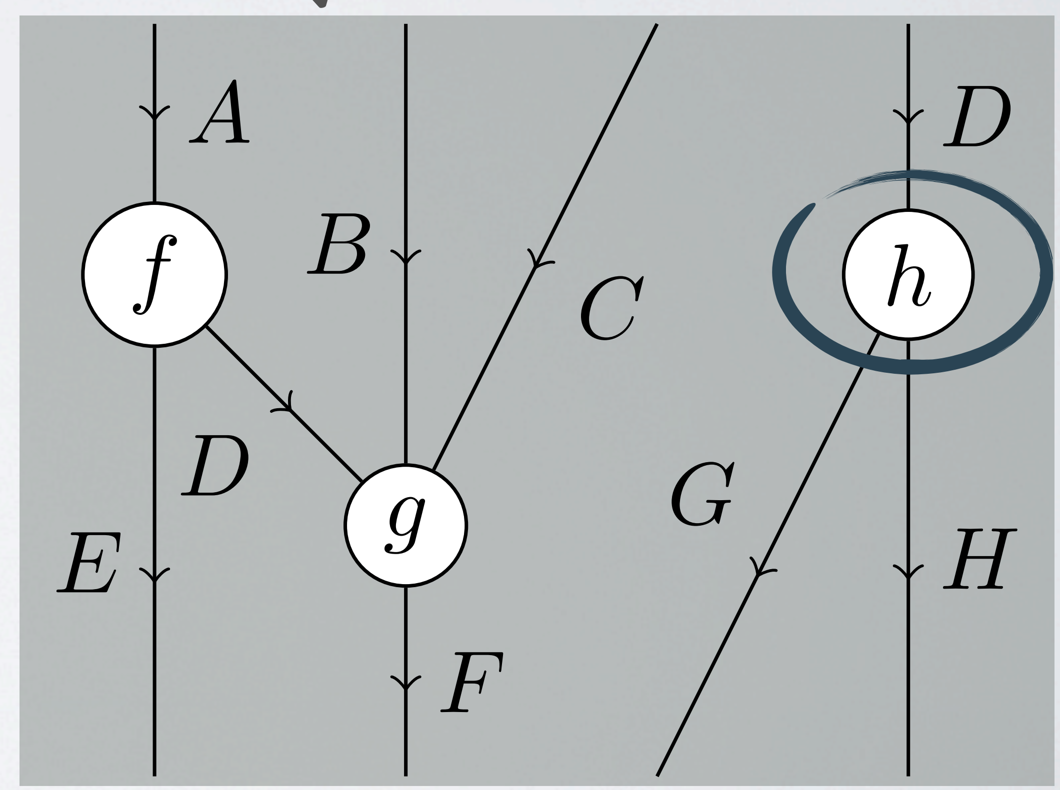
A	$A \otimes B$	$A \otimes B \xrightarrow{f} C \otimes D$	$A \otimes B \xrightarrow{f} C \otimes D \xrightarrow{g} E$



STRING DIAGRAMS FOR MONOIDAL CATEGORIES

$$(E \otimes g \otimes h) \circ (f \otimes B \otimes C \otimes D) = (E \otimes g \otimes G \otimes H) \circ (f \otimes B \otimes C \otimes h)$$





Full treatment: André Joyal and Ross Street's *Geometry of tensor calculus I*. Advances in Mathematics 1991
Hard to find online! :(

IDEA OF THEOREM

- **Theorem**. These symbolic expressions form a (free strict) monoidal category.
- **Theorem**. (Suitably-defined) labelled diagrams form a strict monoidal category.
- **Theorem**. These categories are monoidally equivalent.
  - Notion of diagram *valuation* and *evaluation*
  - Canonical diagram construction

STRING DIAGRAMS FOR MONOIDAL CATEGORIES

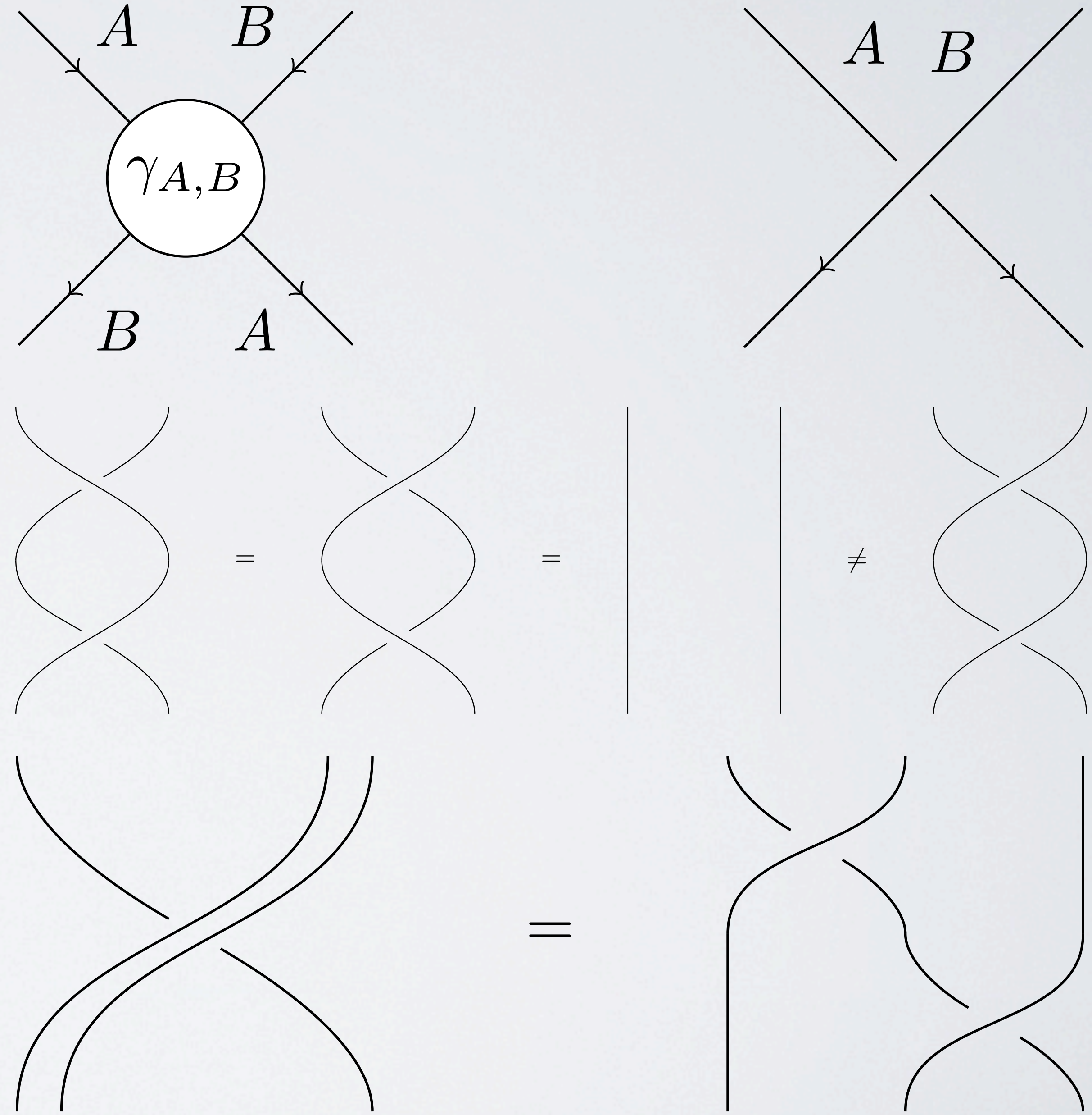
- This gives us:
  - Diagrams are a valid notation
  - Deformations on a diagram preserves valuation in category
  - We can do mathematics using these diagrams

ADDING STRUCTURE, AUGMENTING GRAPHS

- Can add more structure to a monoidal category
  - Can augment graphical language to capture new axioms
  - Some examples...

BRAIDING

- Add a **braiding** natural isomorphism
  - Coherence
  - Eg. category of braids



Full treatment: André Joyal and Ross Street's *Braided tensor categories*. Advances in Mathematics 1993

BRAIDING

- **Theorem**. (Joyal and Street) Free braided monoidal category is equivalent to category of labelled braids.
- **Theorem**. (Reidemeister) Manipulating braid diagrams corresponds exactly to isotopy on braided strings in 3-space.
- **Corollary**. (Joyal and Street) Two expressions are isomorphic iff the underlying braids are the same.

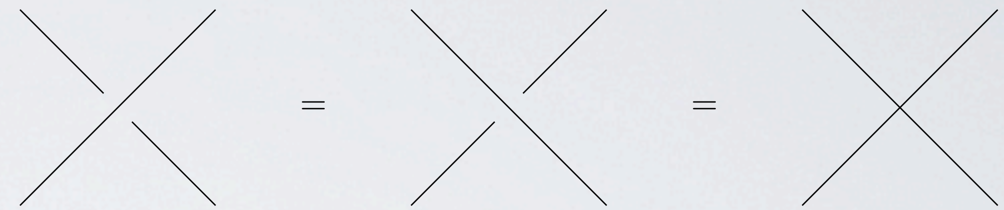
BRAIDING

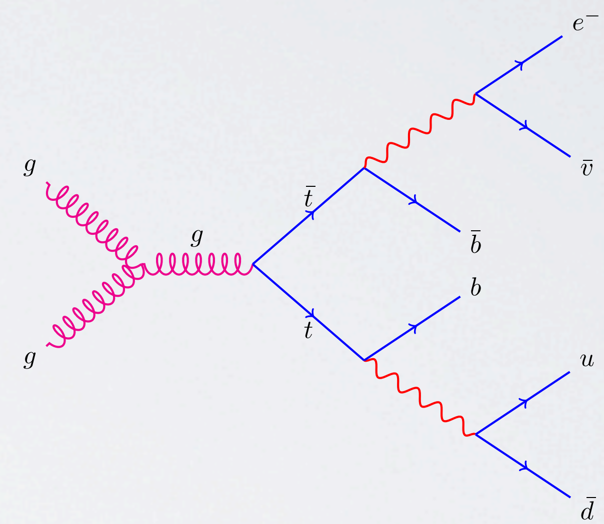
- We see some horrendous braiding isomorphisms...

- ...are just the identity! We've saved a lot of chalk.

COMPACT CLOSED

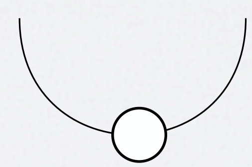
- A **symmetry** is a self-inverse braiding
  - Sets; Feynman diagrams; ...



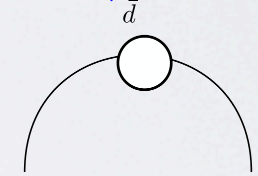


dream.inf.ed.ac.uk/projects/quantomatic/talks/cambridge-2010-2x2.pdf

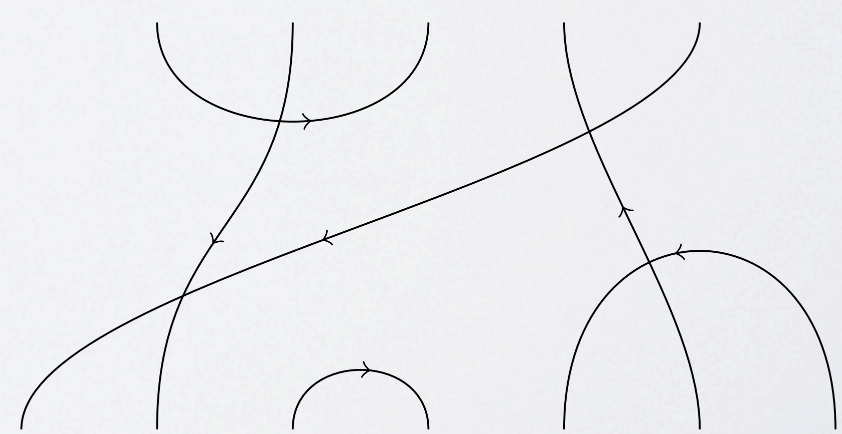
- A **compact closed** category is symmetric with *(right) duals*
  - Vector spaces; ...



$e_A : A \otimes A^* \to I$



$d_A : I \to A^* \otimes A$



RIBBON

- A **twist** in a braided monoidal category is a natural isomorphism

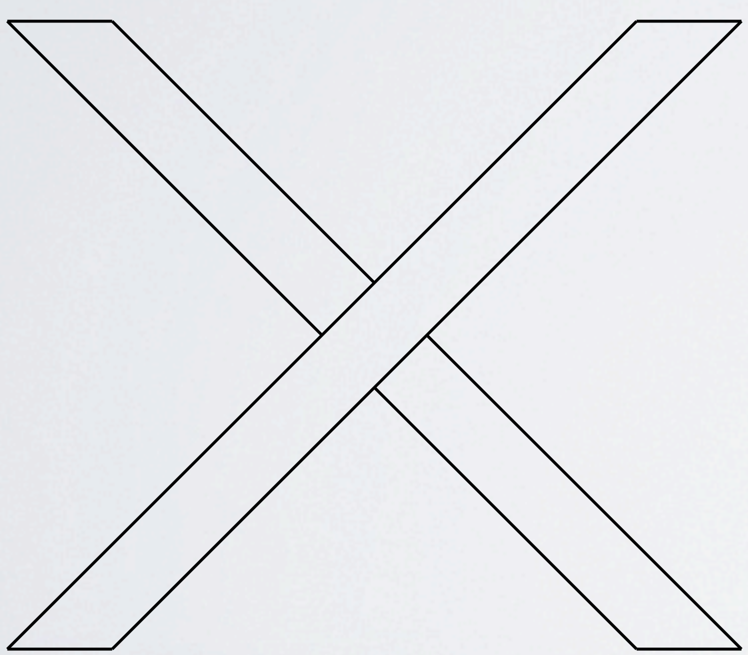
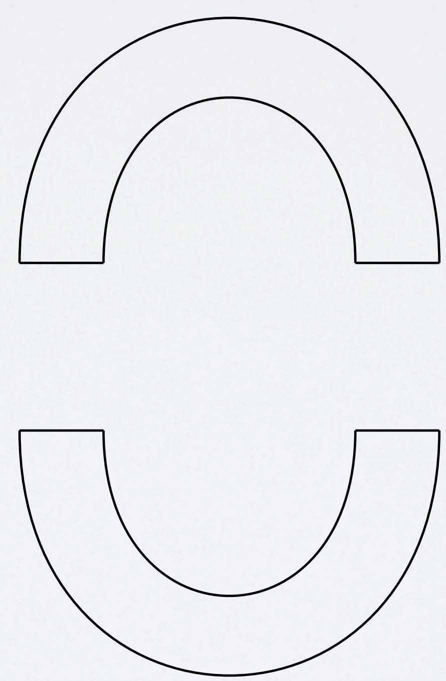
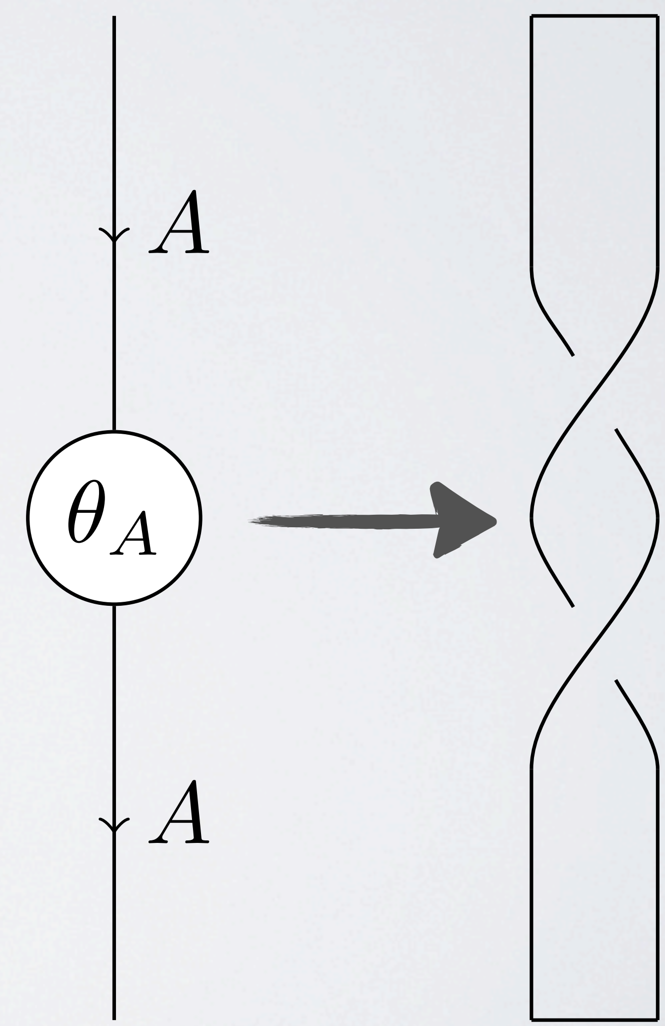
$$\theta_A : A \to A$$

coherent with the braiding.

- A **tortile** or **ribbon category** is a braided monoidal category with a dual for each object and a twist (plus axioms)

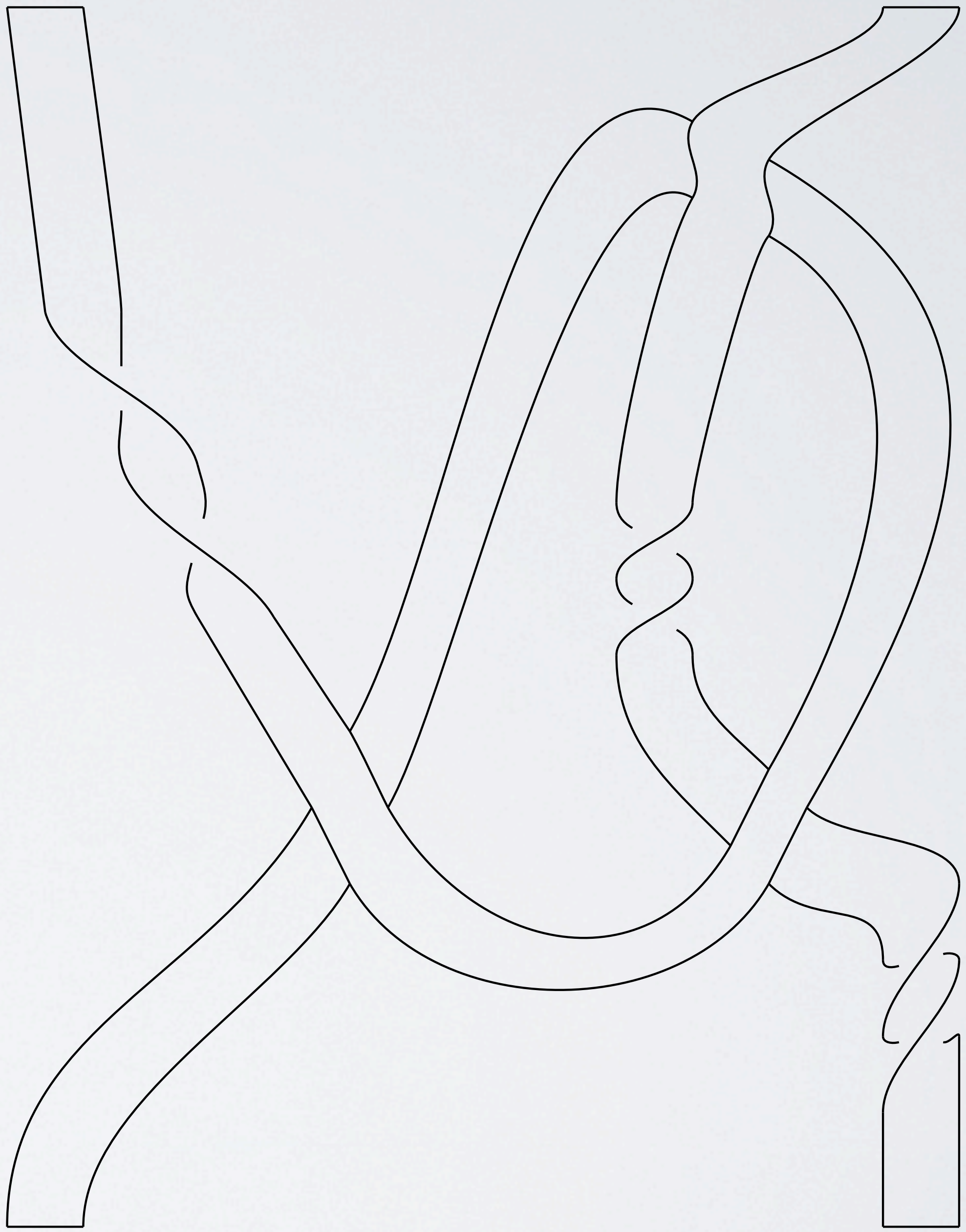
Full story: Mei Chee Shum's *Tortile tensor categories*. Journal of Pure and Applied Algebra 1994

RIBBON

Braiding	Duals	Twist
		

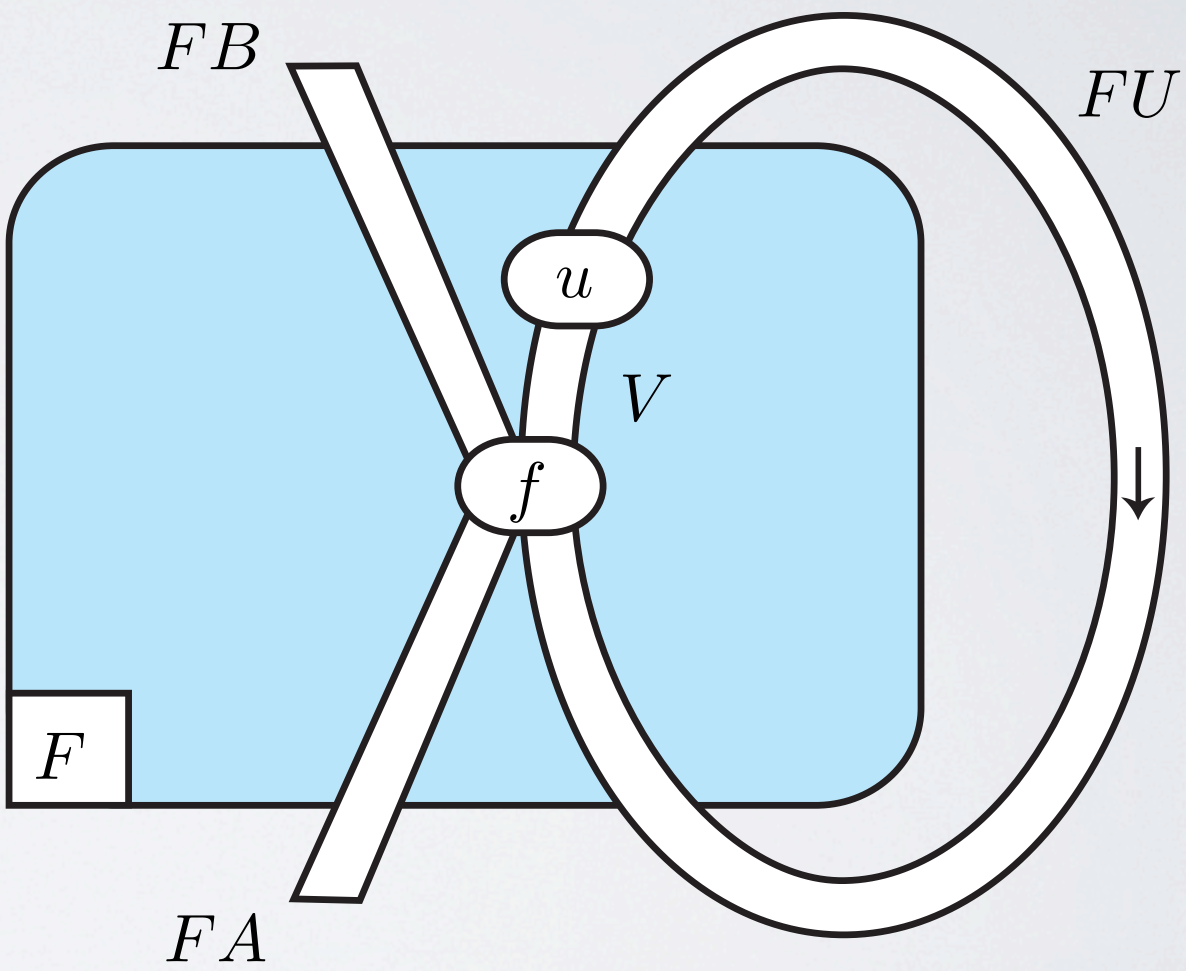
RIBBON

- These compose to form pictures
  - Like ribbon tangles in 3-space!
    - (Missing a lot of detail again...)
  - Useful for knot invariants, quantum protocols



EVEN FURTHER

- Functorial boxes
- Higher categories



Melliès' *Functorial boxes on string diagrams*. Computer Science Logic 2006 pps.jussieu.fr/~mellies/papers/functorial-boxes.pdf
Instructional videos: The Catsters' *String diagrams*. youtube.com/view_play_list?p=50ABC4792BD0A086

WHAT I'M DOING

- Similar motivations:
  - Formalise graphical language people already use for argument
  - Not a monoidal category!

GAME SEMANTICS

- Model computational environment as interaction "games"
  - *Player* or *proponent* is system, *opponent* is environment
  - *Game* is alternating sequence of moves
- Games model types
- *Strategies* for player model terms

Many introductions around. Here's slides from a recent talk by Guy McCusker at LI2012:
li2012.univ-mrs.fr/media/talk19/mccusker-lectures.pdf

GAME SEMANTICS

$$\mathbb{N}$$

- Example:

$$\begin{array}{lll} q & & O \\ 3 & & P \end{array}$$

GAME SEMANTICS

- Example:

$$\begin{array}{ccccc}
\mathbb{N} & \to & \mathbb{N} & \\
 & & q & O \\
q & & & P \\
3 & & & O \\
 & & 4 & P
\end{array}$$

GAME SEMANTICS

- Arrow games, $A \multimap B$
  - Two games in parallel
  - Roles reversed roles on left
  - Moves interleaved
    - Interleaving is a *schedule*

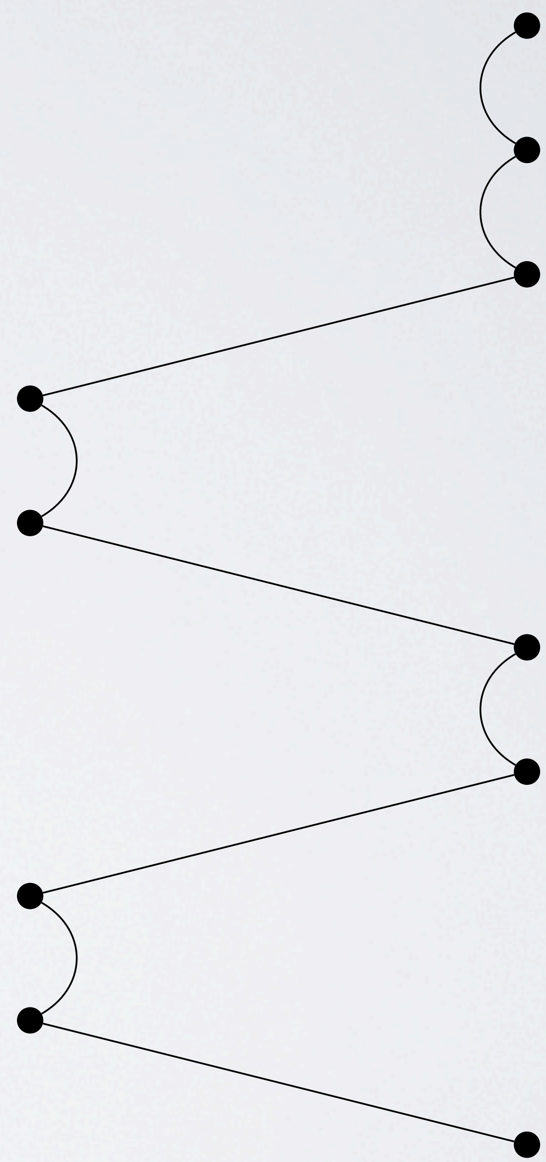
SCHEDULES

- Originally definition combinatorial in nature
  - Can be thought of as binary strings
  - Or "collectively surjective" function pairs, or order relations
- Composition is highly combinatorial (and tricky)
- Associativity is difficult to establish

Good stuff here: Russ Harmer, Paul-André Melliès and Martin Hyland's *Categorical combinatorics for innocent strategies*. LICS 2007
pps.jussieu.fr/~mellies/papers/lics2007-categorical-combinatorics.pdf

SCHEDULES

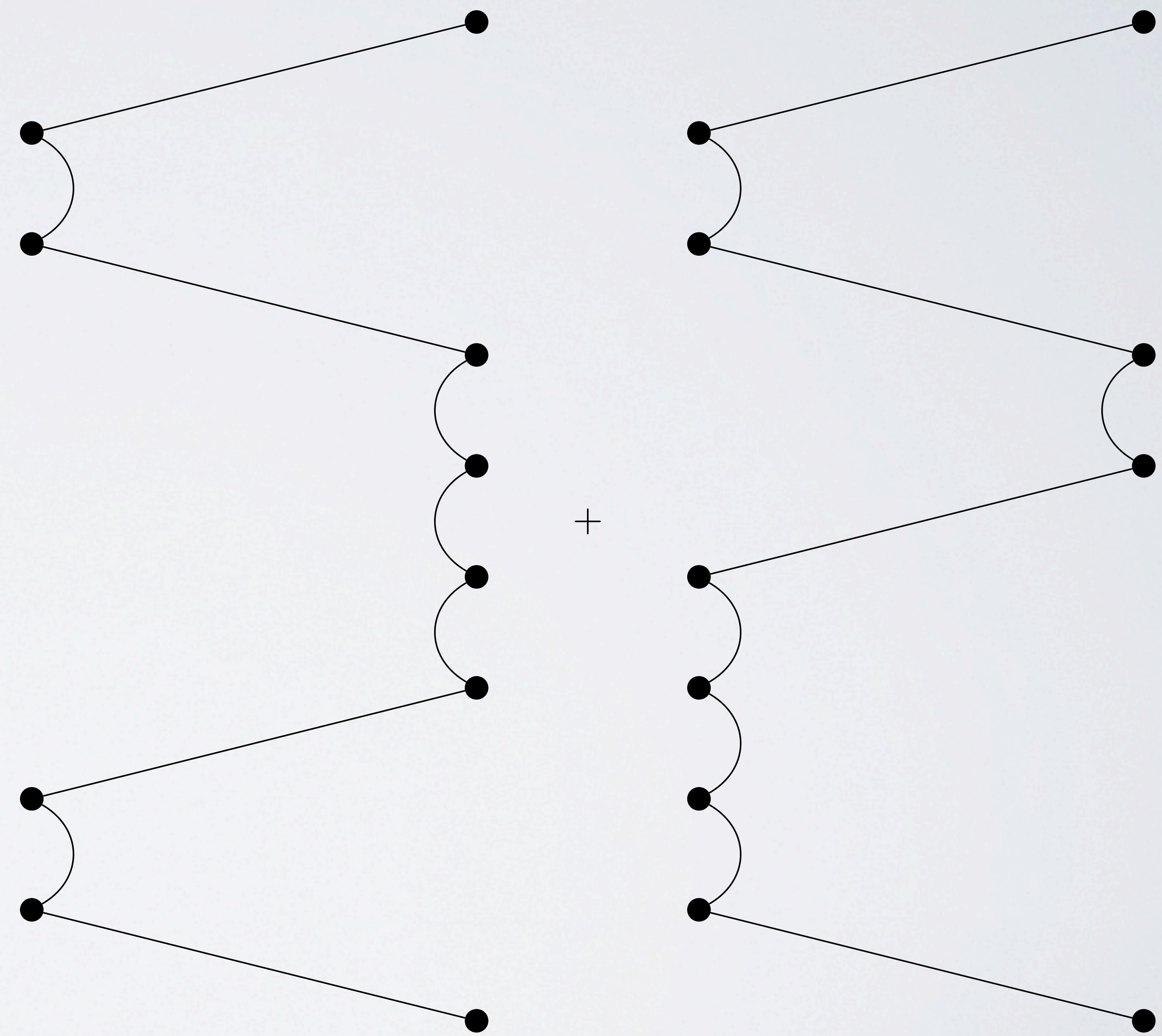
- Composition and associativity are tricky to do by hand
- People tend to use pictures



Guy McCusker, John Power and Cai Wingfield's *A graphical foundation for schedules*. MFPS 2012, ENTCS

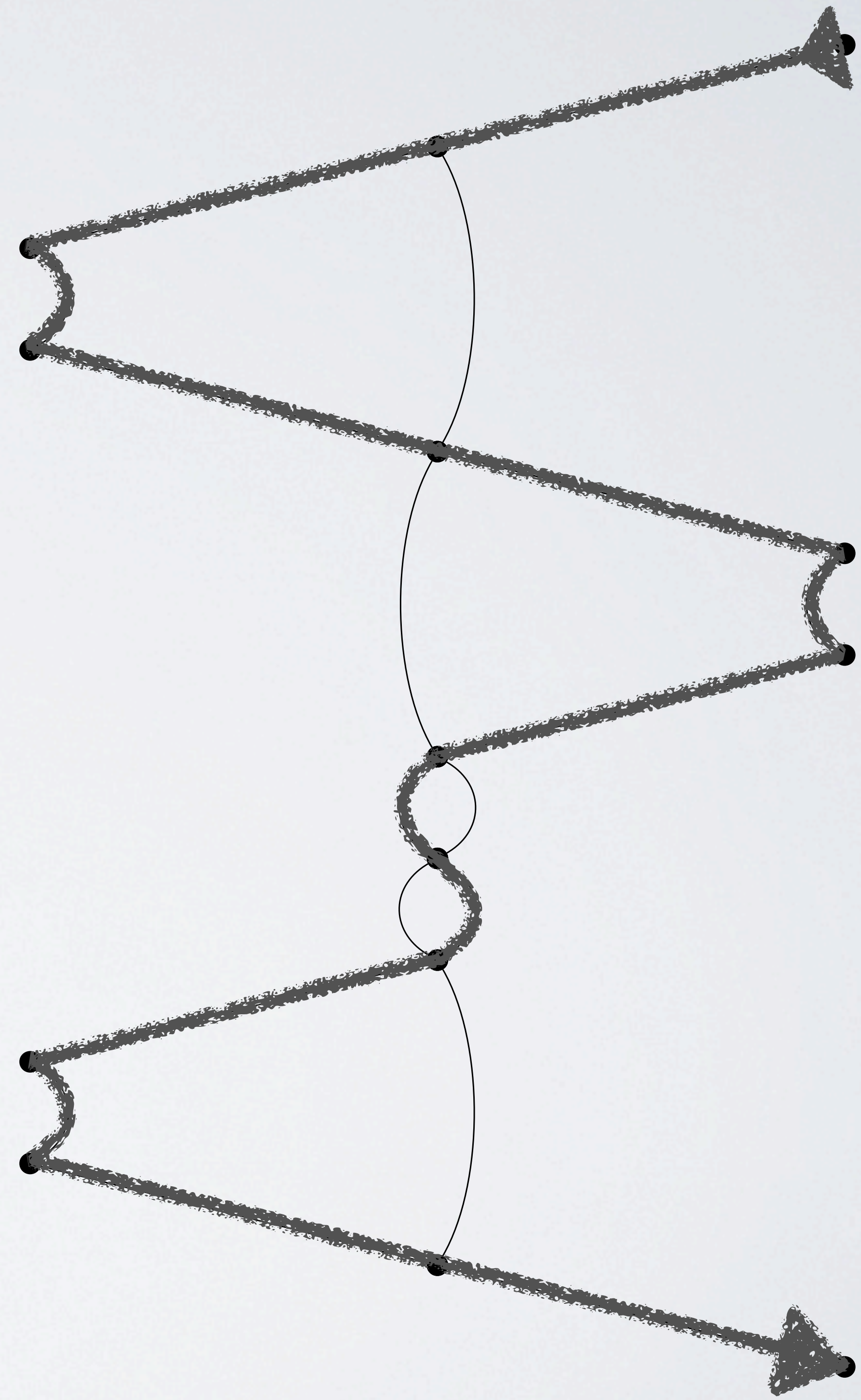
SCHEDULES

- Composition:



SCHEDULES

- Composition:
  - Glue schedules
  - Trace path through all nodes



SCHEDULES

- Composition:
  - Glue schedules
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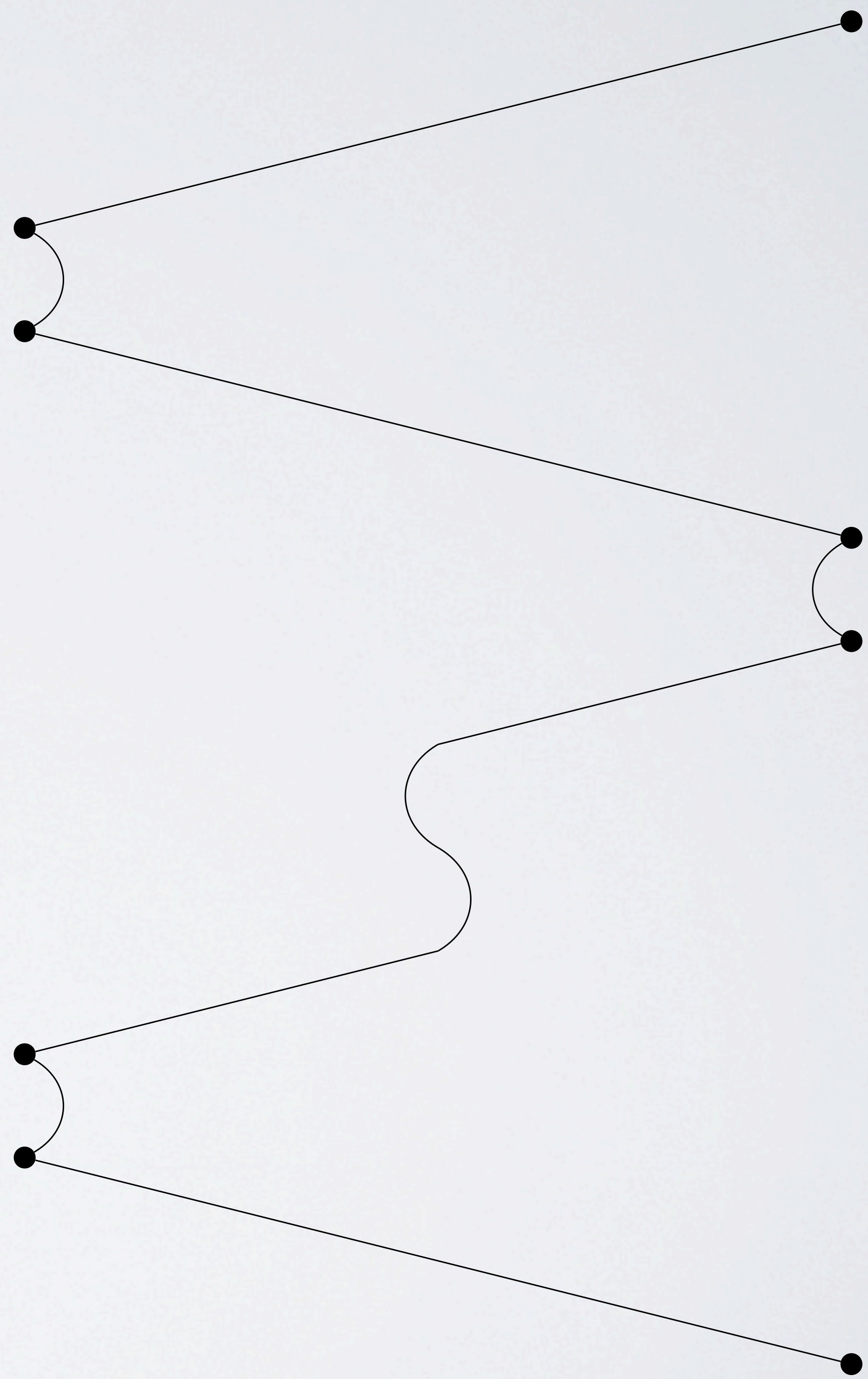
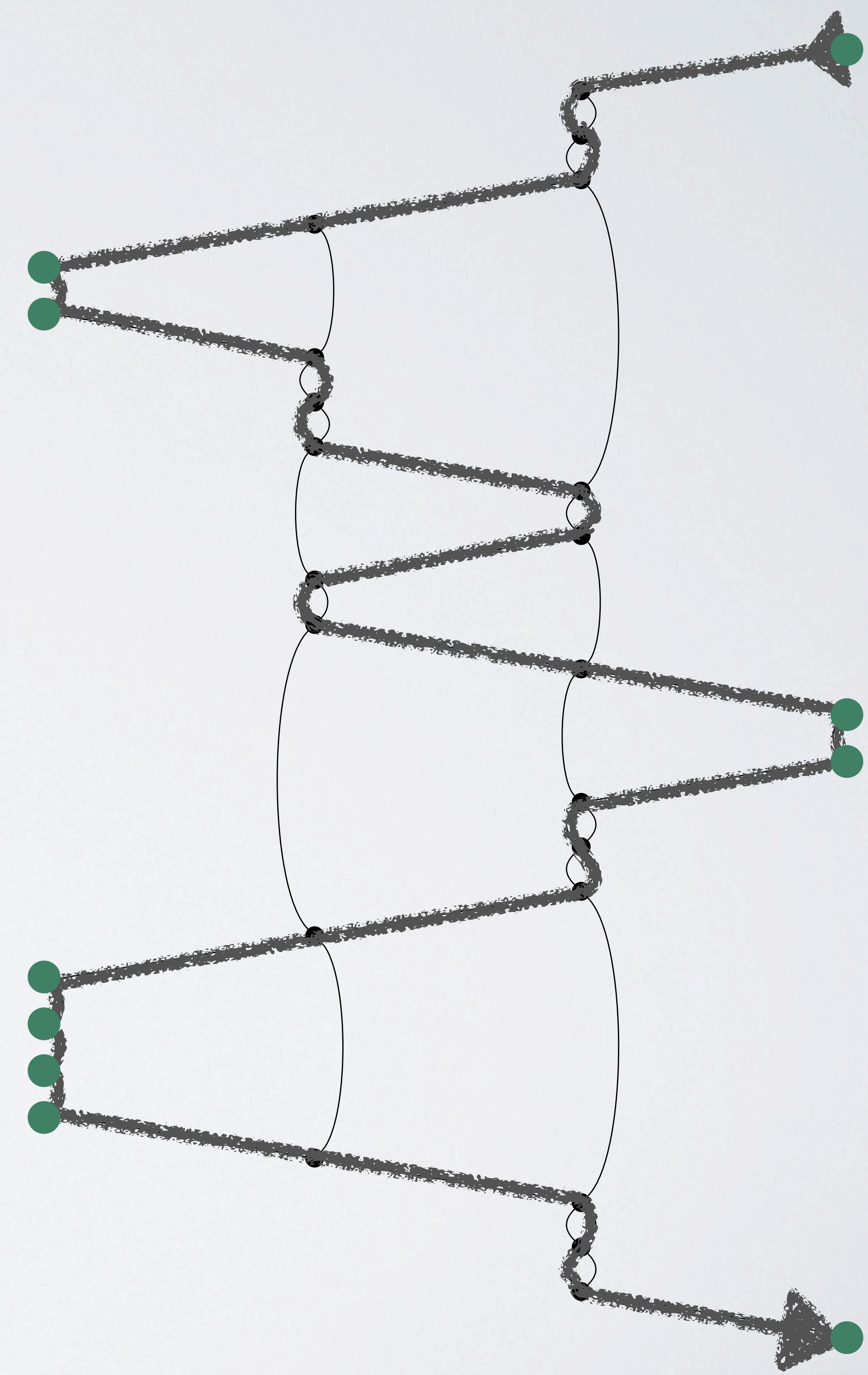


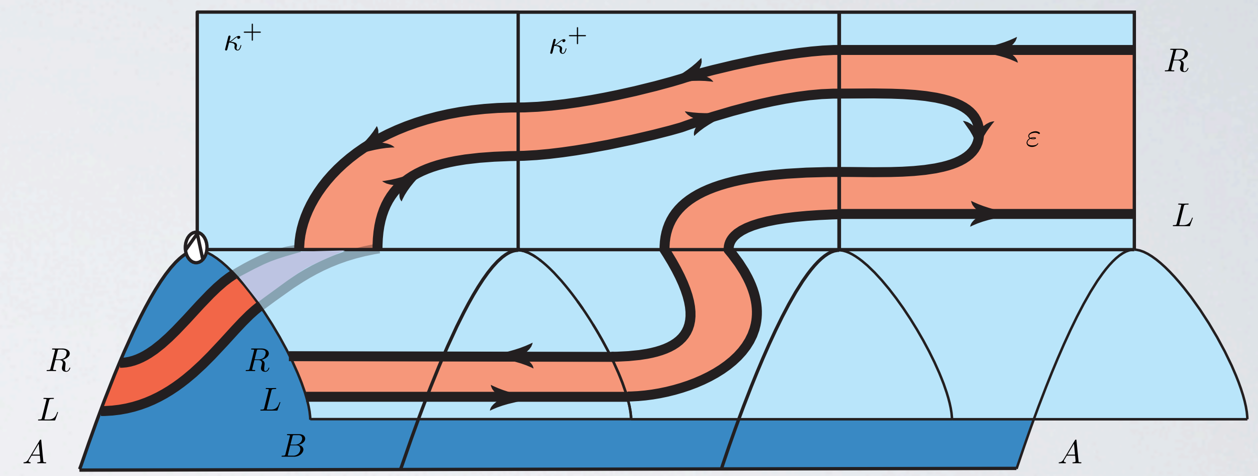
SCHEDULES

- Associativity becomes easy!
  - "Juxtaposition in the plane is associative"



USES, RESEARCH

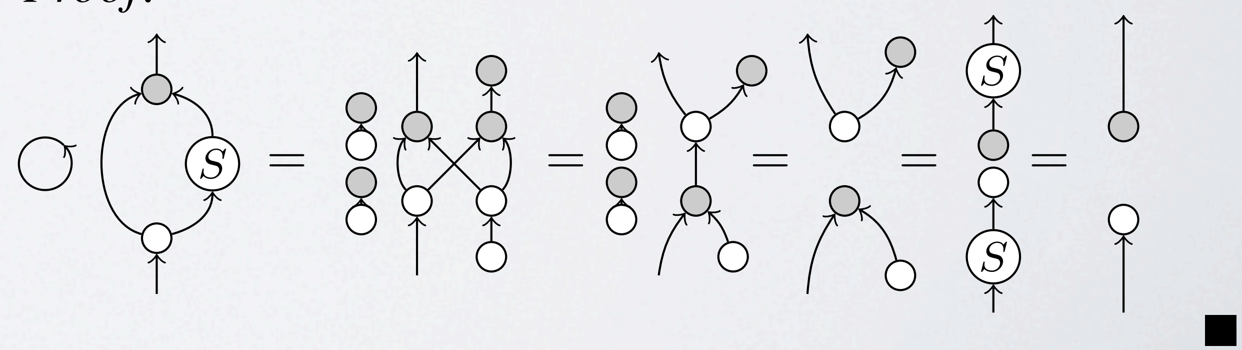
- Things I heard about at Logic and Interaction 2012
  - Melliès' *Tensorial logic*
  - Coecke, Duncan, Kissinger and Wang: categorical quantum mechanics



$$\neg\neg A \otimes \neg\neg B \vdash \neg\neg(A \otimes B)$$

pps.jussieu.fr/~mellies/tensorial-logic.html

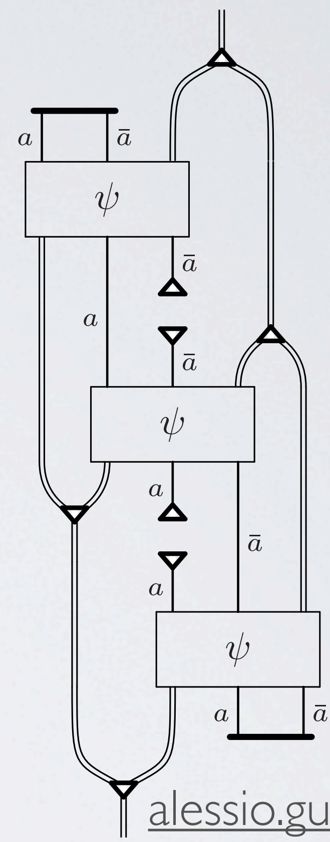
Theorem 3.6: Strong complementarity ⇒ complementarity.
Proof:



arxiv.org/abs/1203.4988

USES, RESEARCH

- More things (off the top of my head):
  - Girard's *Proof nets*
  - Guglielmi's *Atomic flows for deep inference*
  - Lafont's *Algebraic theory of boolean circuits*



alessio.guglielmi.name/res/cos/

$$\begin{array}{|cccc|}\hline 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ \hline \end{array}$$

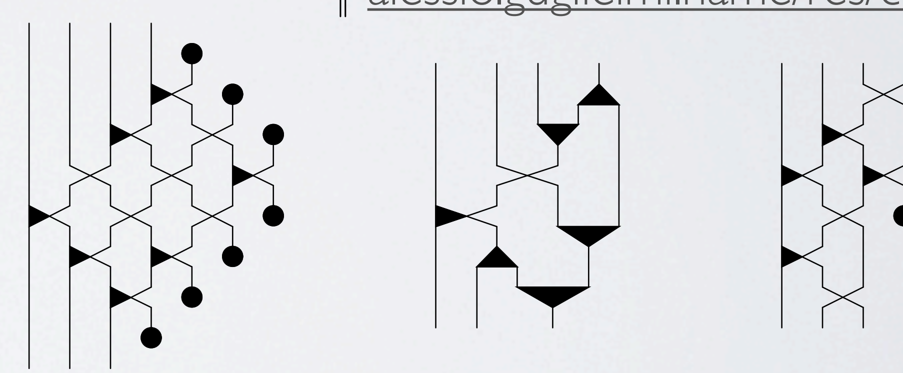


Figure 23: The canonical forms of a matrix in $\mathbf{L}(\mathbb{Z}_2)$

iml.univ-mrs.fr/~lafont/pub/circuits.pdf

Want to draw nice string diagrams for LaTeX? Check out Aleks Kissinger's cross-platform GUI front-end to TikZ, *TikZiT*: tikzit.sourceforge.net/