Graphical Foundations for Dialogue Games

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Mathematical Foundations Group
University of Bath

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Queen Mary University of London

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    - Pointers for backtracking.

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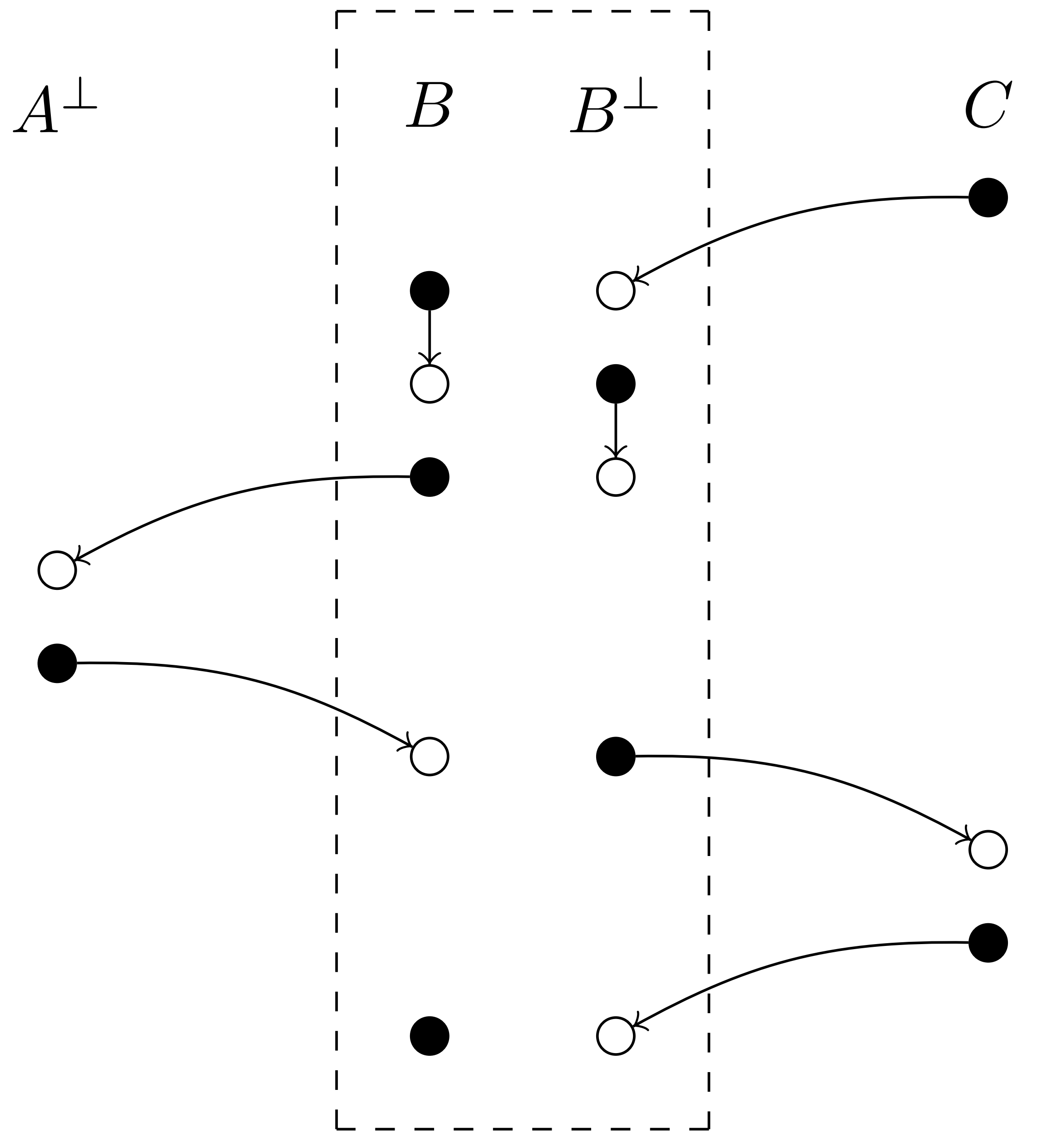
- Let games be *given by* their diagrams, rather than the correspondence being informal or suggestive.
- Let intuitive arguments become proofs in terms of the definitions.
- Give examples of such arguments for categorical properties of games.

Which diagrams to characterise?



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$$
\begin{array}{cccccc}
 & \sigma & & & \tau & \\
A & \multimap & B & B & \multimap & C \\
 & & & & & c_1 \\
 & & & b_1 & & \\
 & & b_1 & & & \\
 & & b_2 & & & \\
 & & & b_2 & & \\
 & & \vdots & \vdots & & \\
 & & & b_k & & \\
 & & b_k & & & \\
a_1 & & & & &
\end{array}
$$

Which diagrams to characterise?

$(\mathbb{N}$	$\Rightarrow$	$\mathbb{N}$	$\Rightarrow$	$\mathbb{N})$	$\Rightarrow$	$\mathbb{N}$	
						q	O
				q			P
q							O
1							P
		q					O
		1					P
				n			O
						n	P

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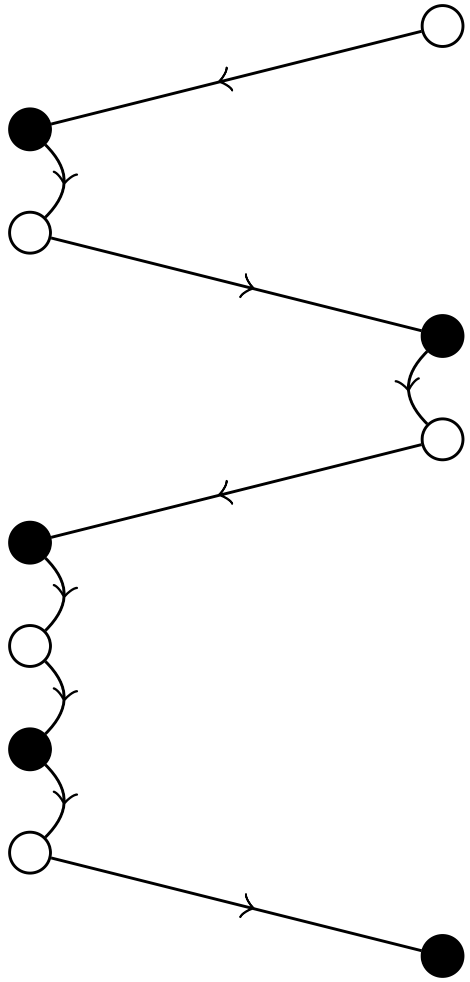
  - n-*Interleaving graphs* have nodes arranged in *n* vertical lines.

Schedules for $\multimap$

Schedules for ⊸



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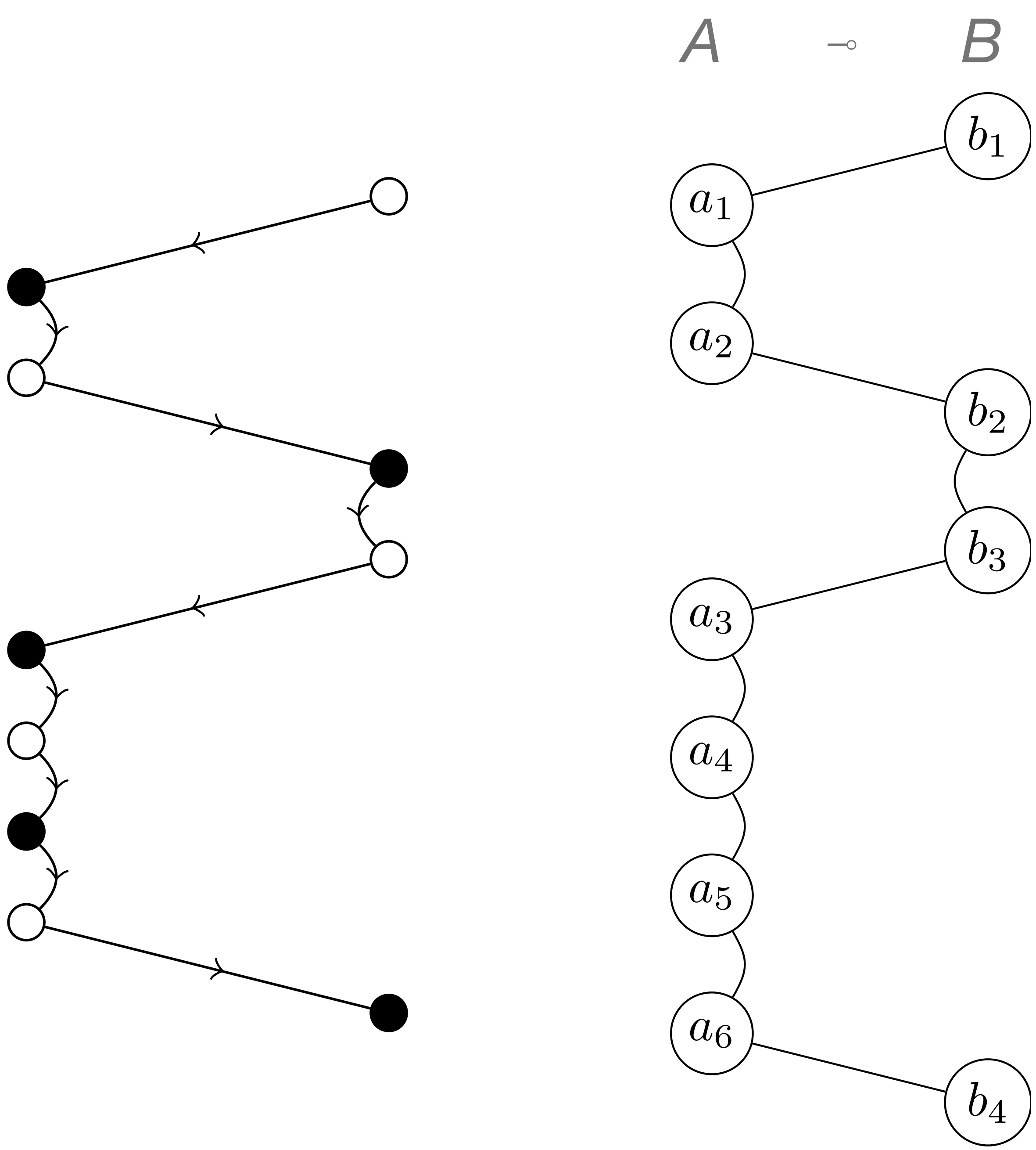

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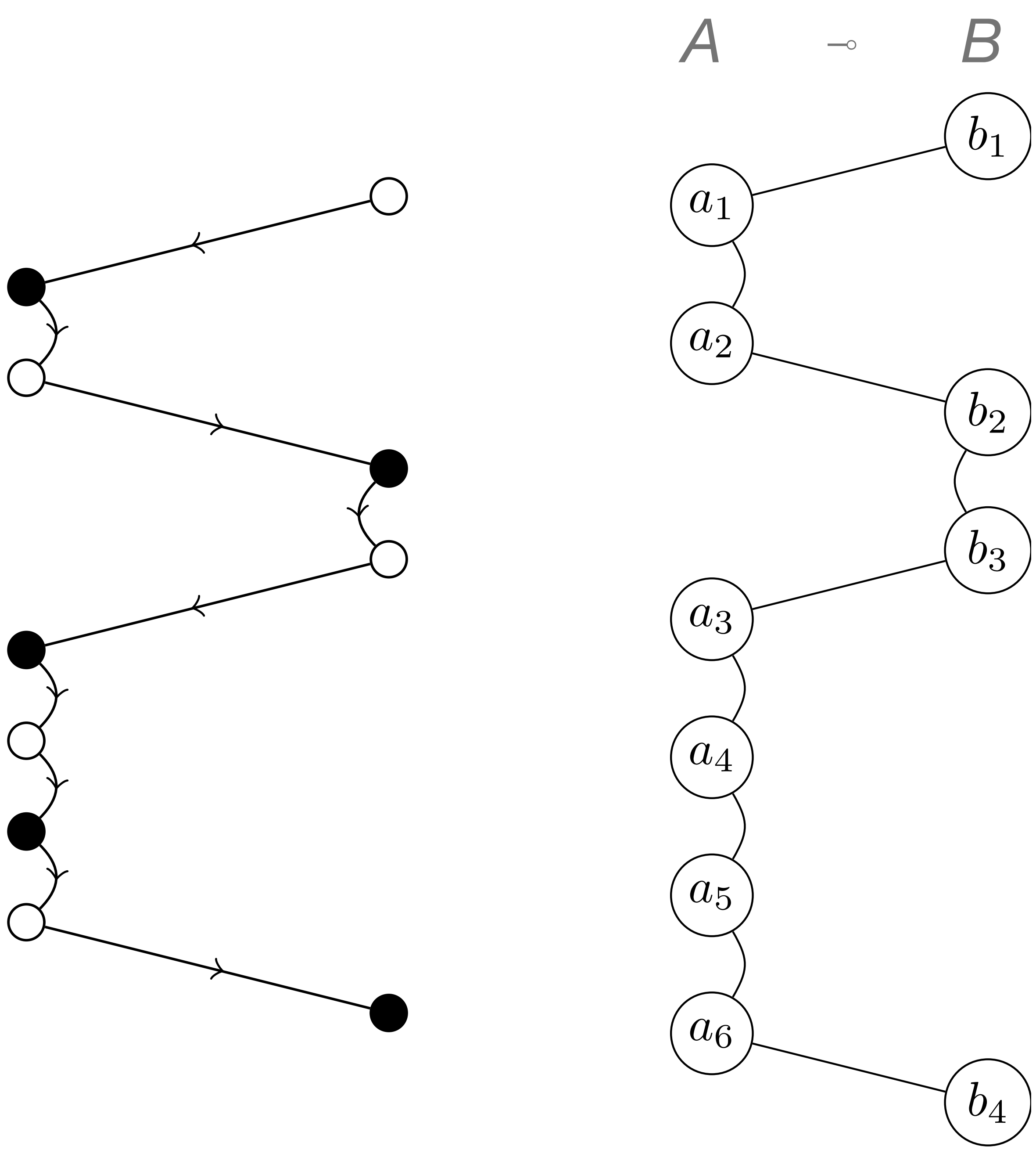

- Given games A and B, the game $A \multimap B$ is that of all positions (S, a, b) such that:

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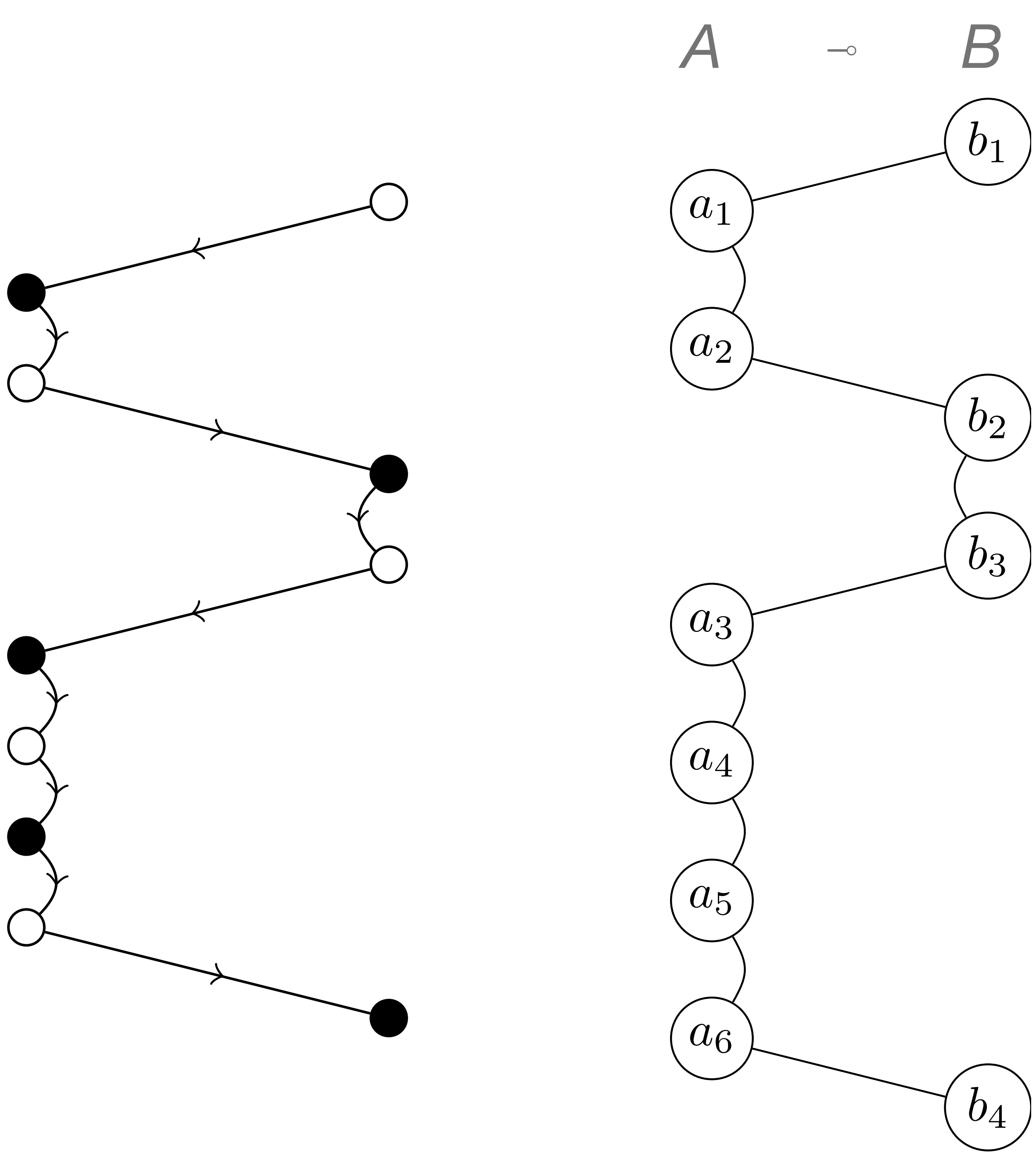

- Given games A and B, the game $A ⊸ B$ is that of all positions (S, a, b) such that:
  - $S : m \to n$.

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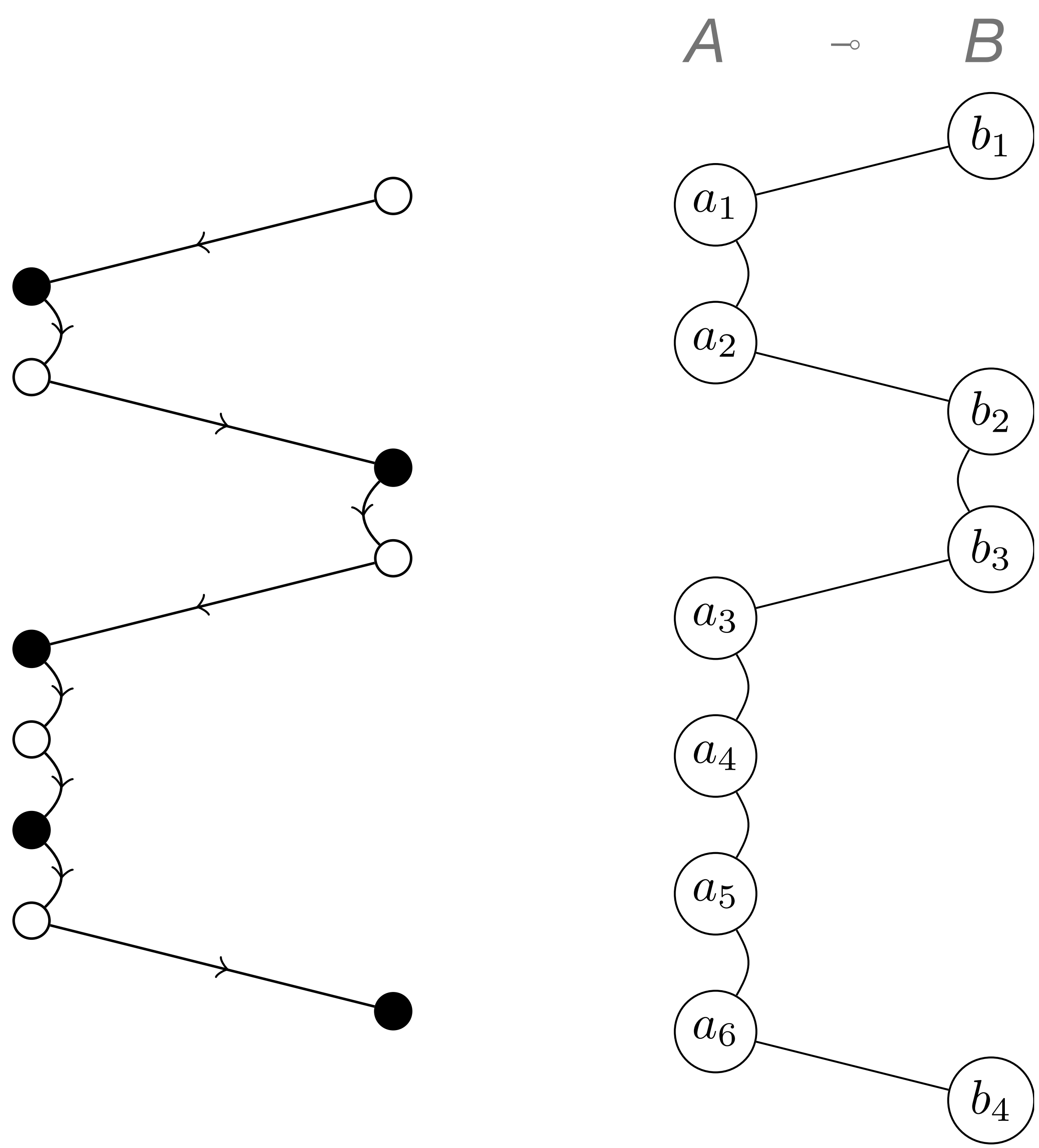

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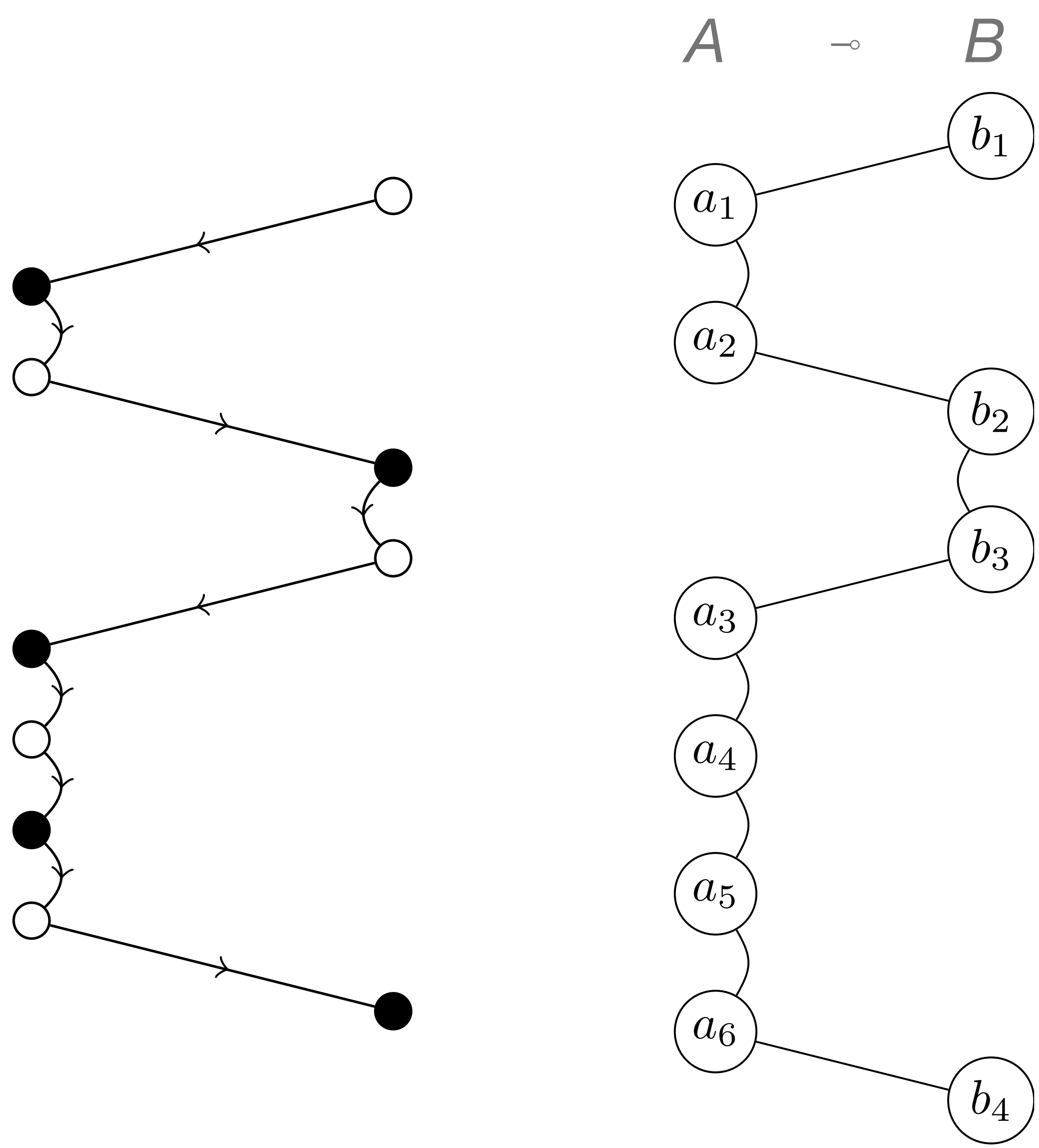

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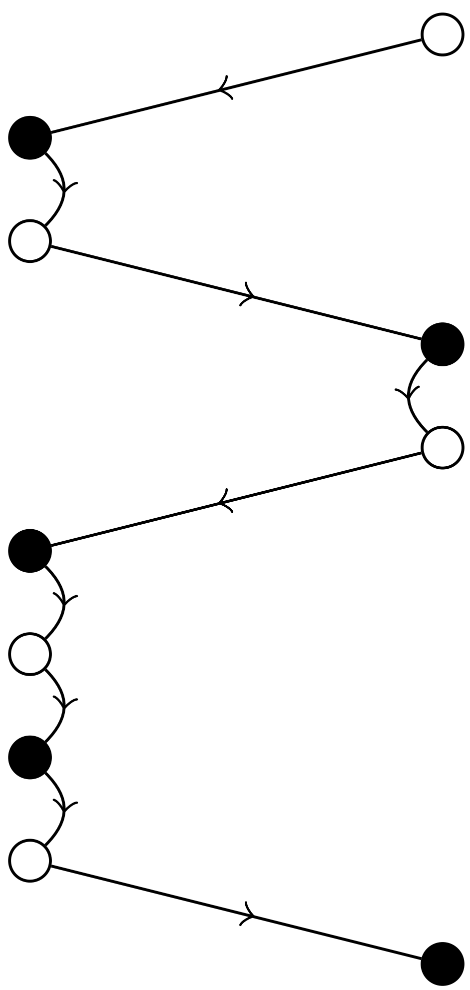
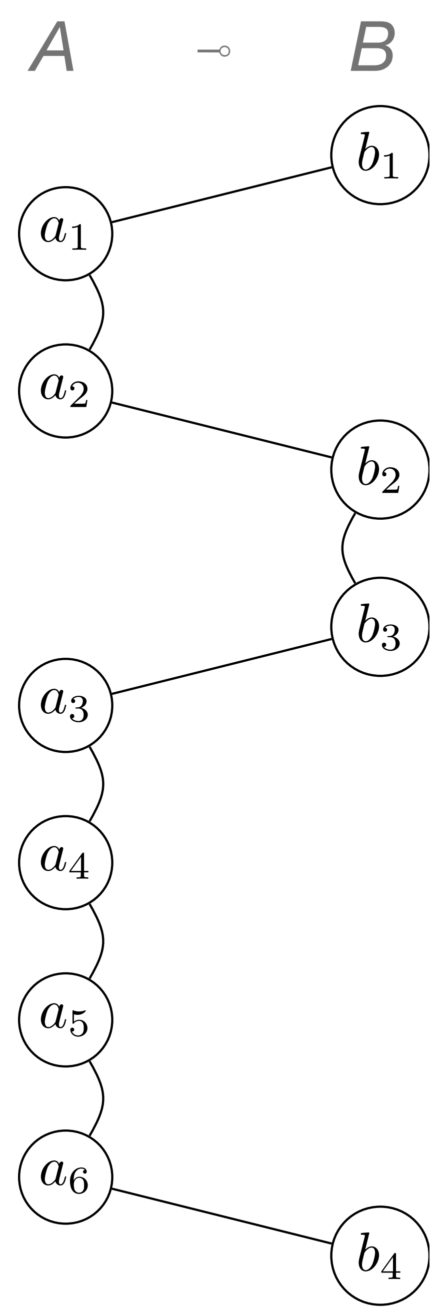
Schedules for $\multimap$





- Given games A and B, the game $A \multimap B$ is that of all positions (S, a, b) such that:
  - $S : m \to n$.
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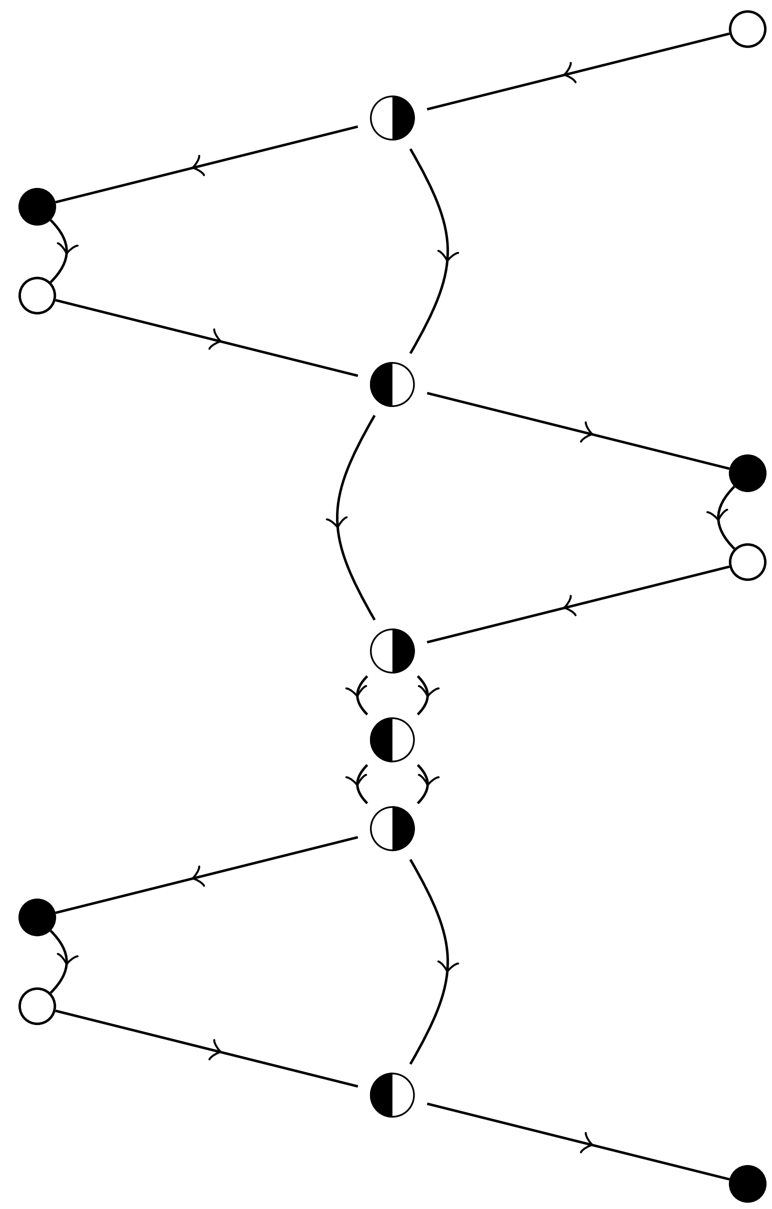
- Predecessor given by truncation.

Composition of schedules



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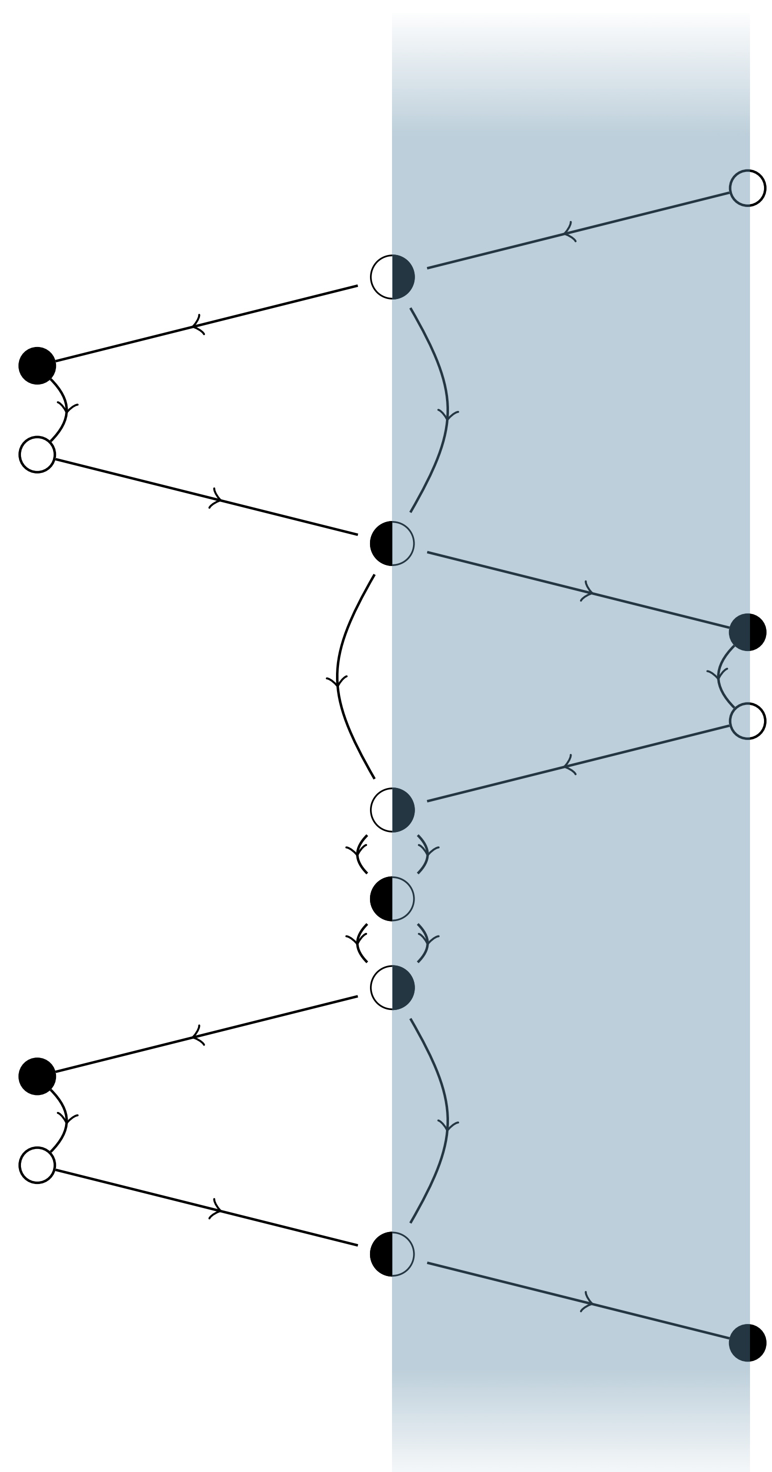

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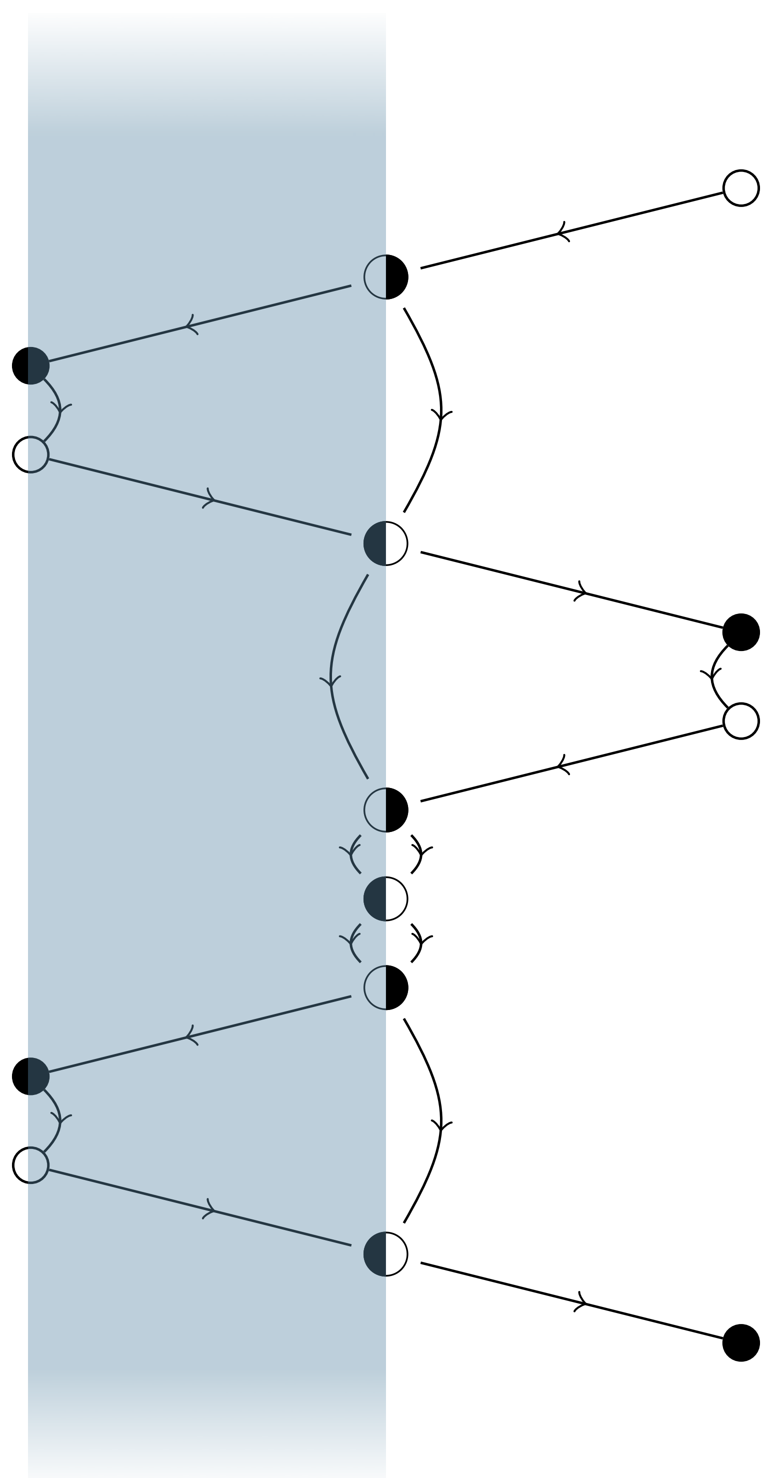

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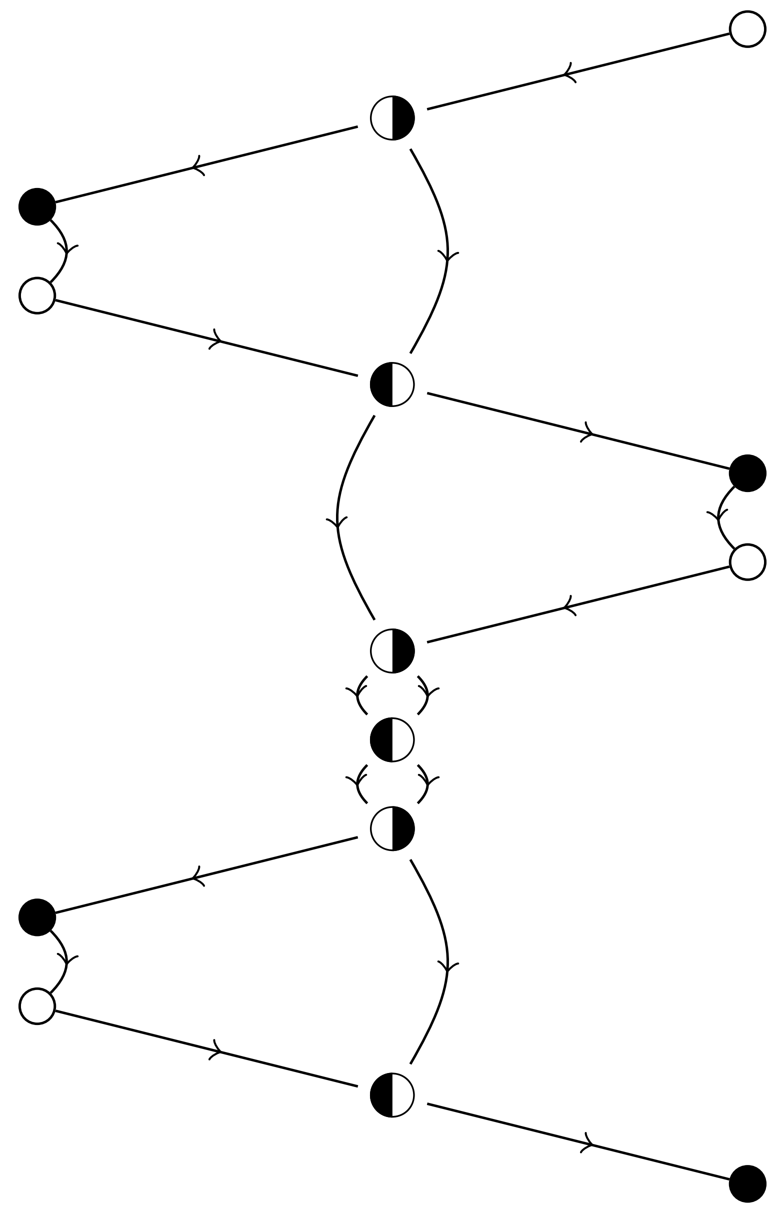

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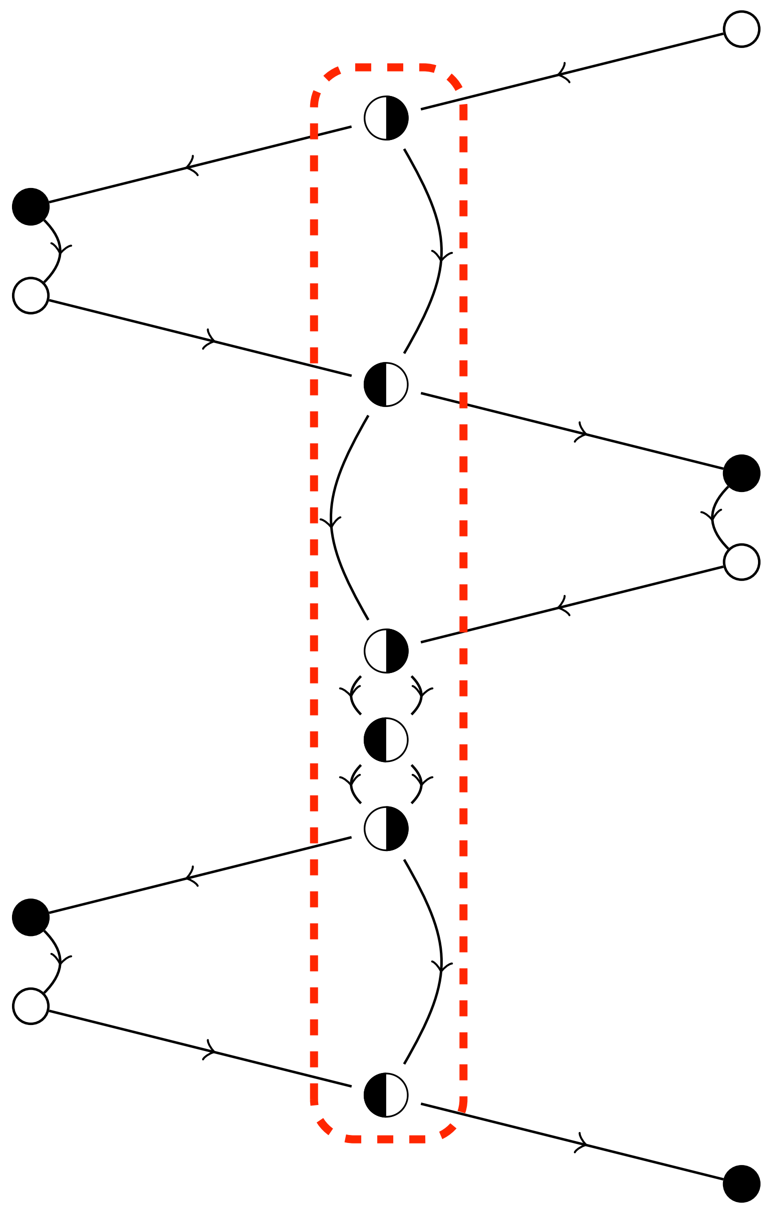

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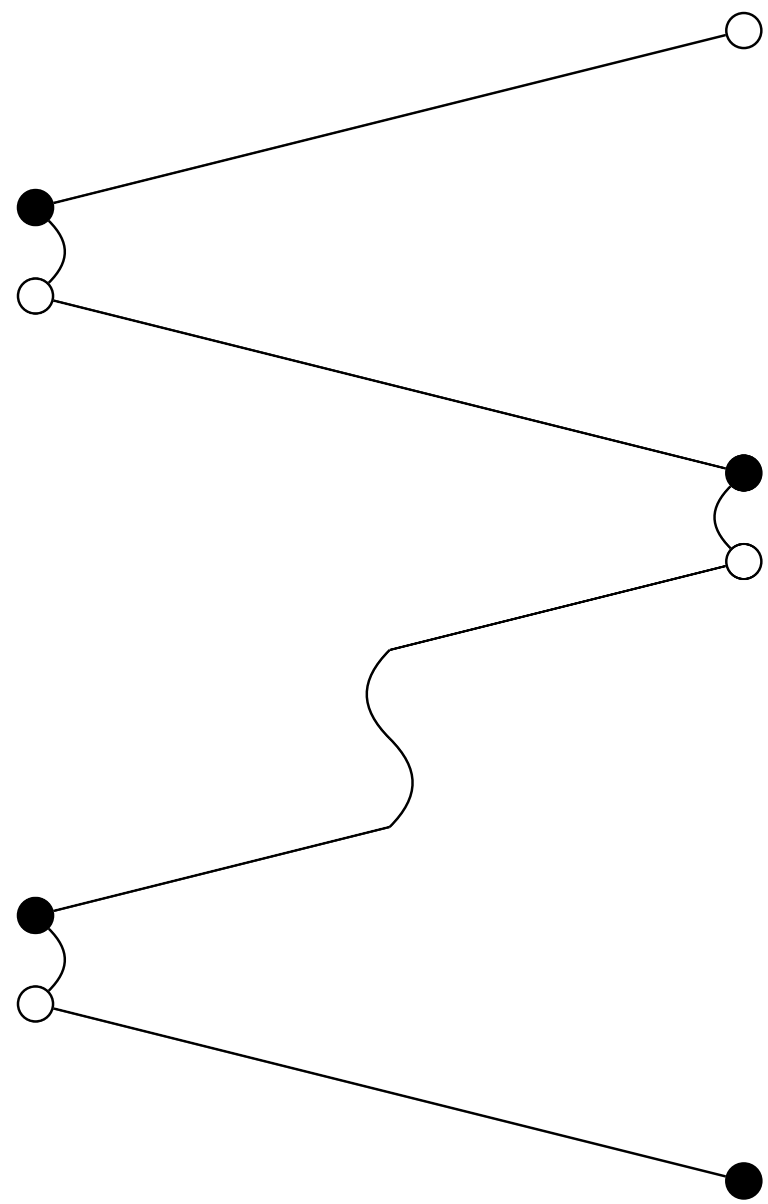

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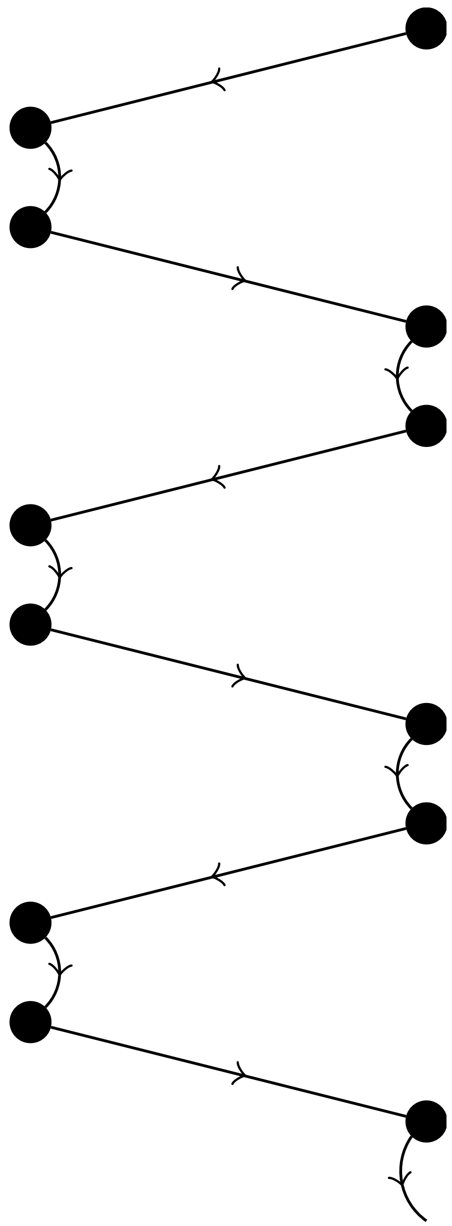

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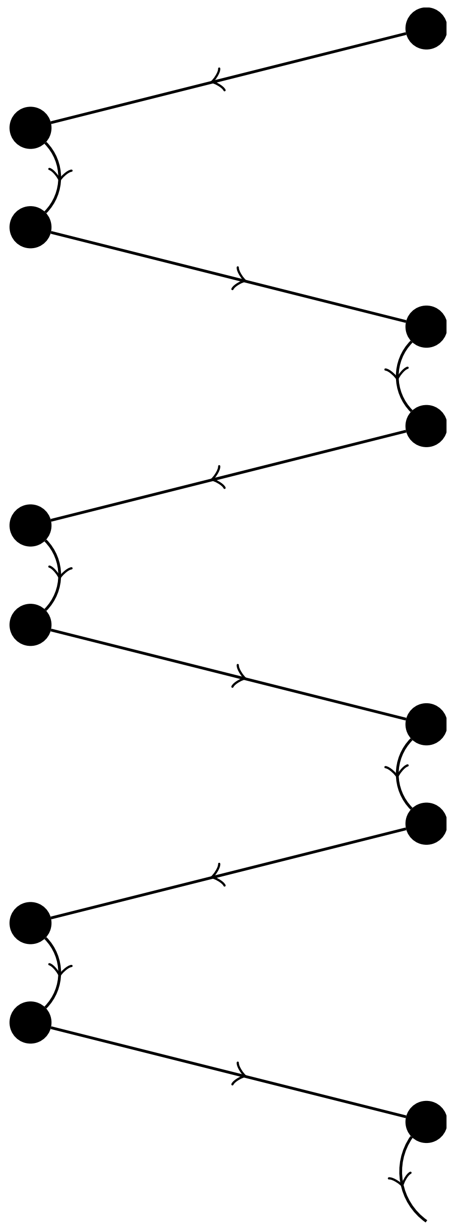

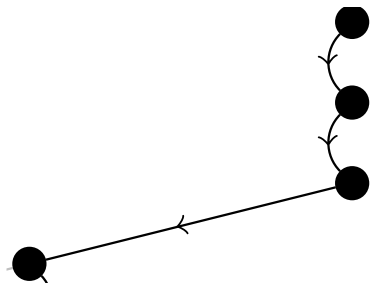

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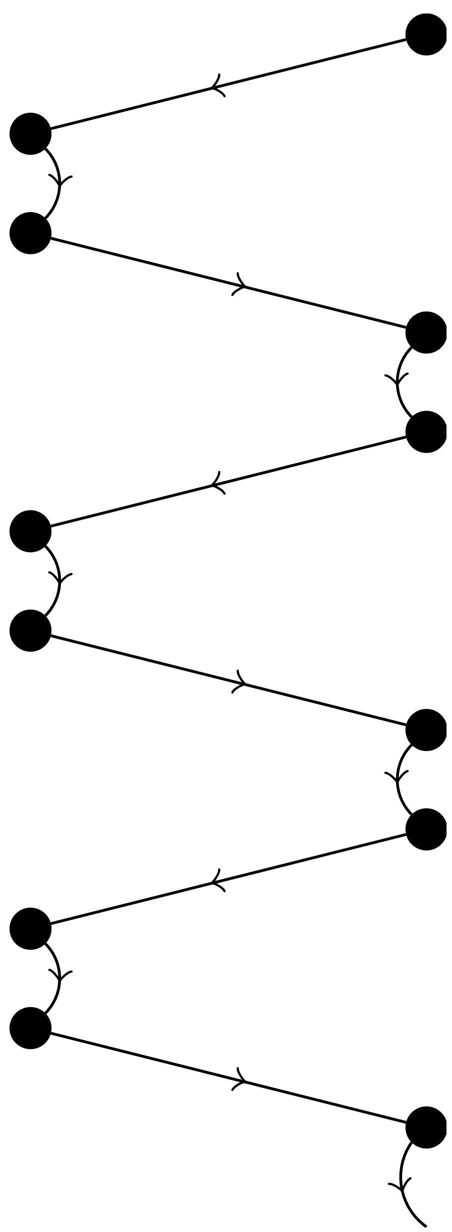

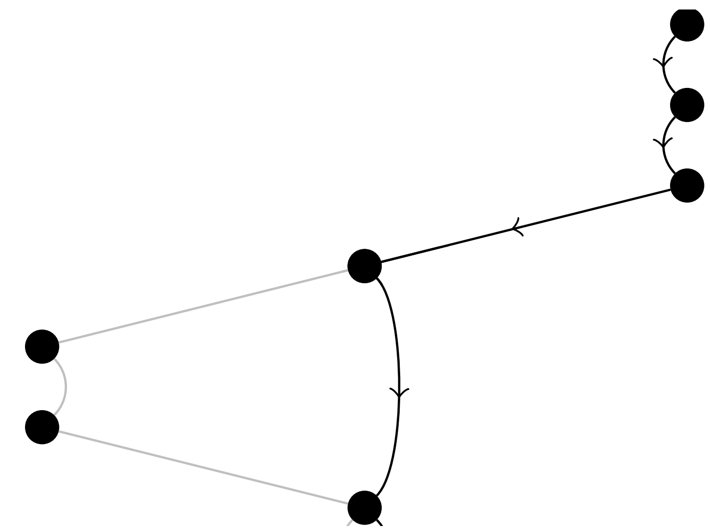

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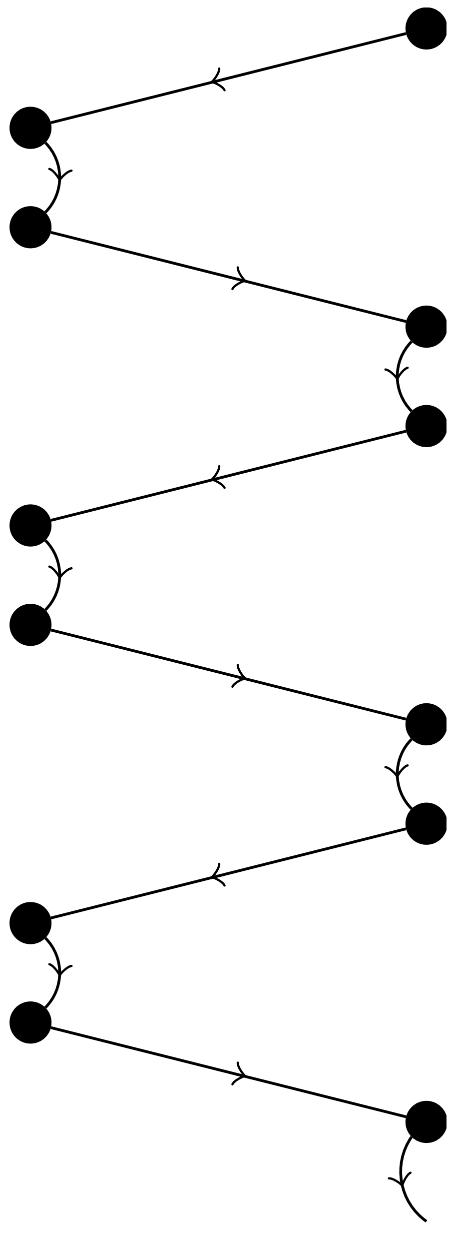

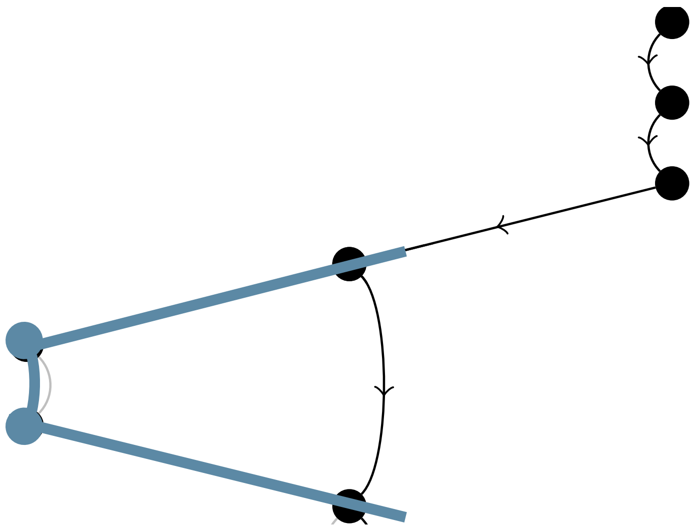

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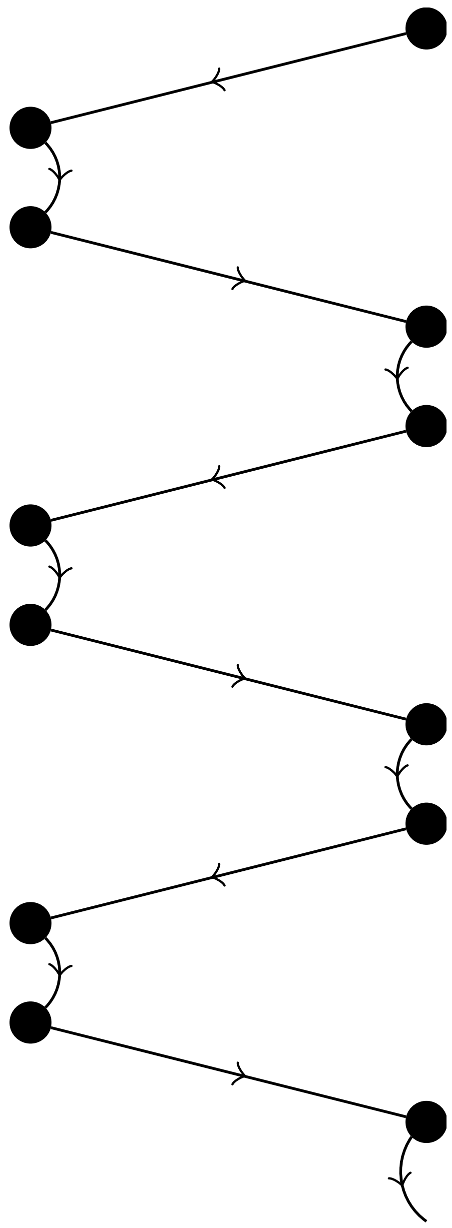

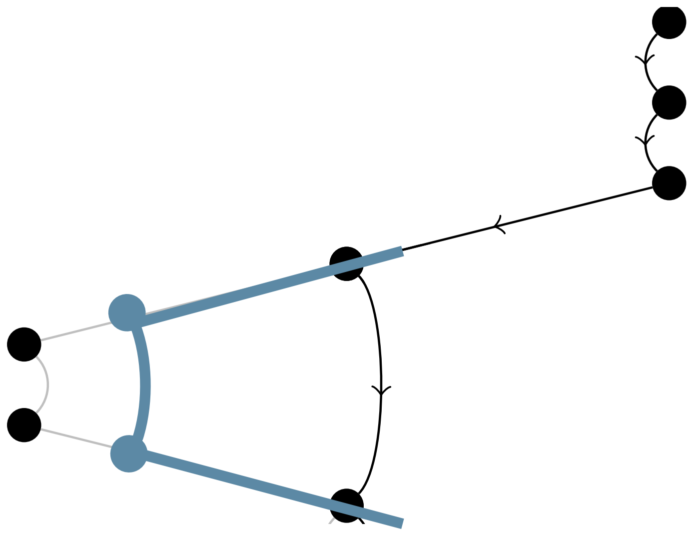

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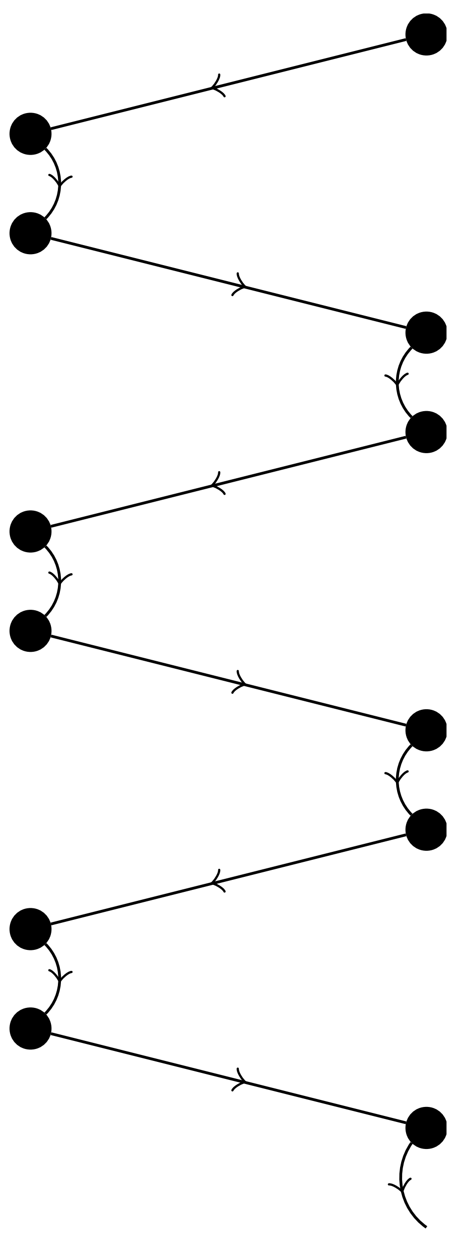

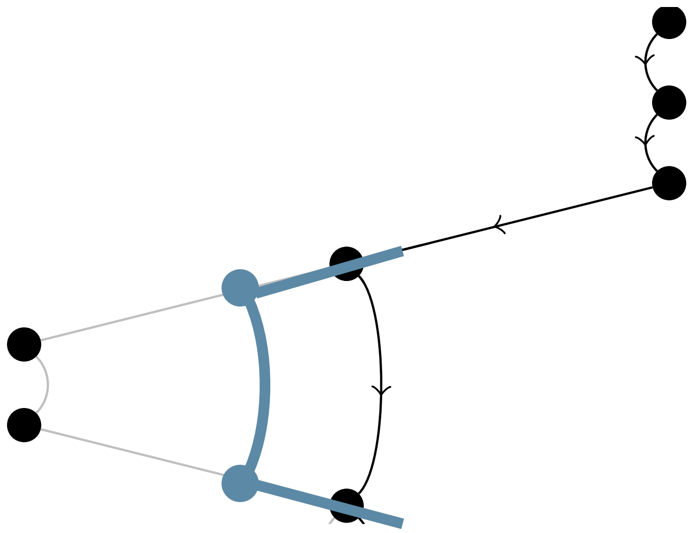

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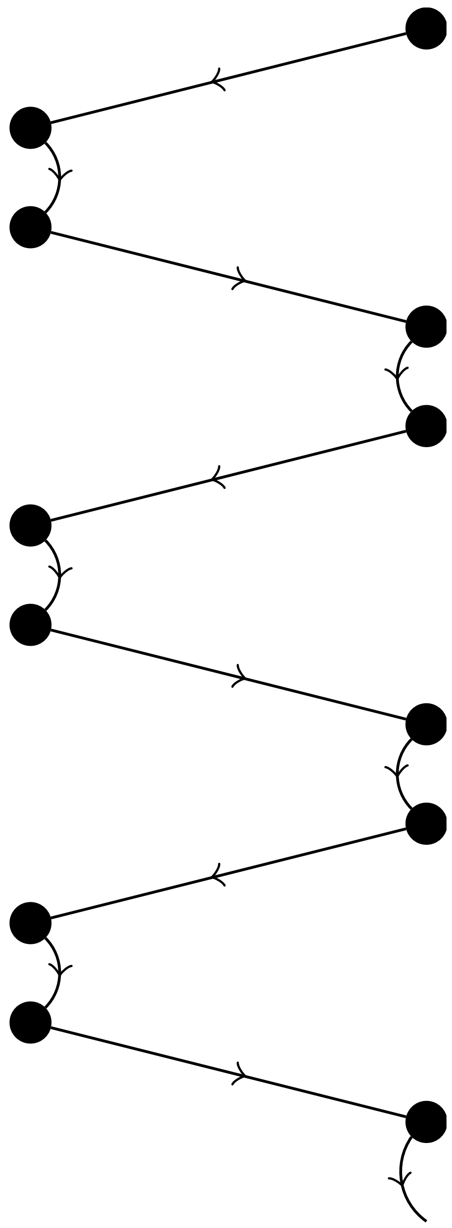

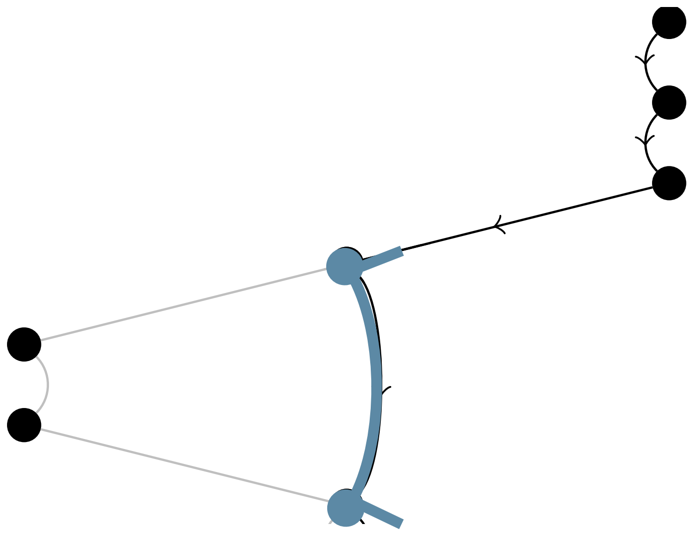

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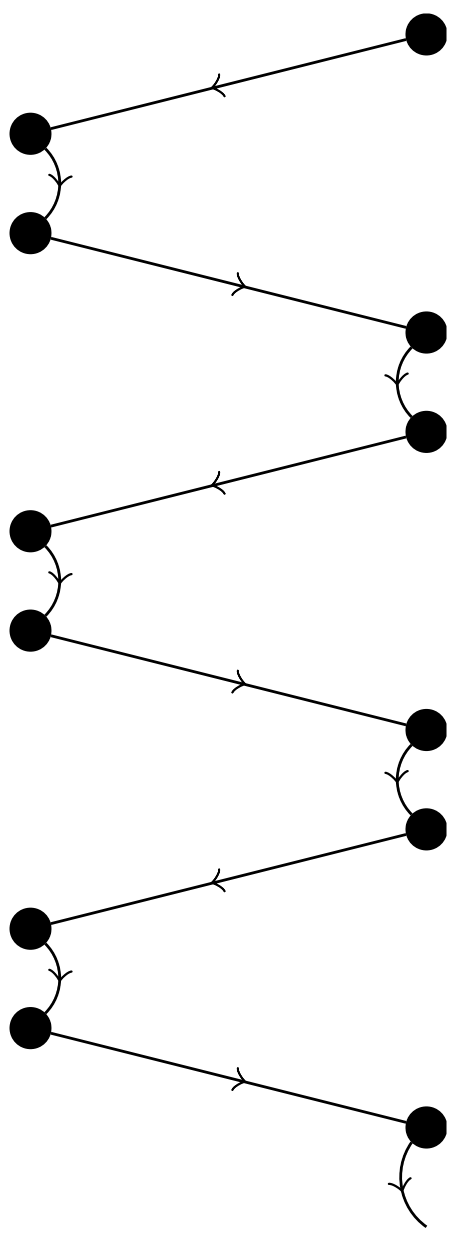

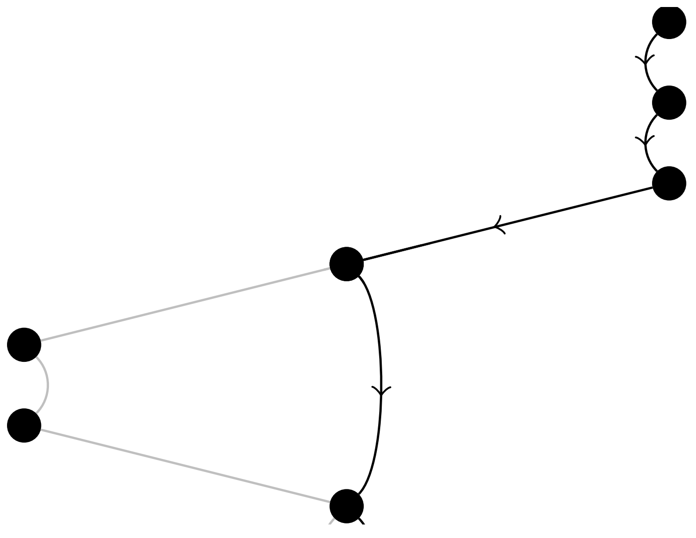

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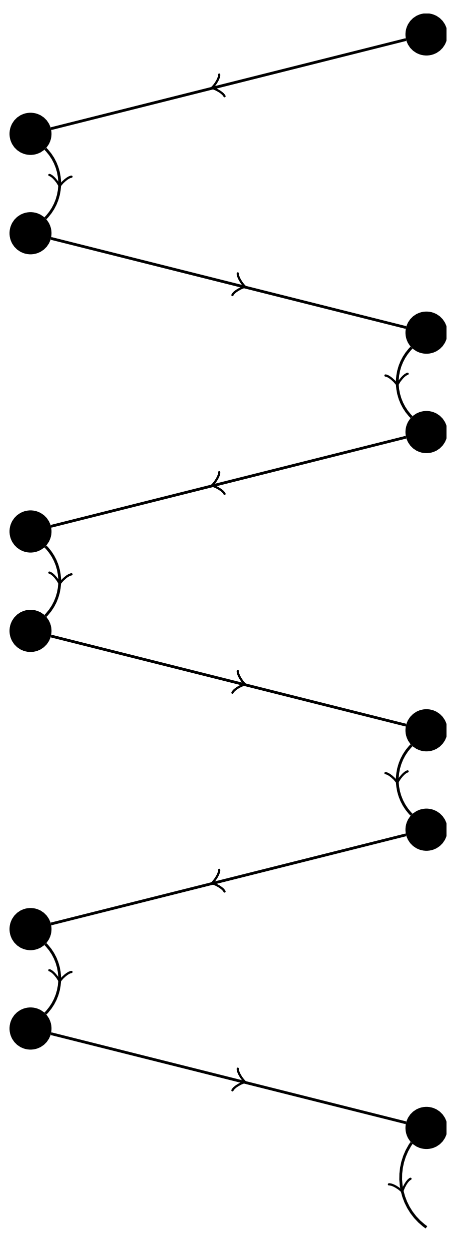

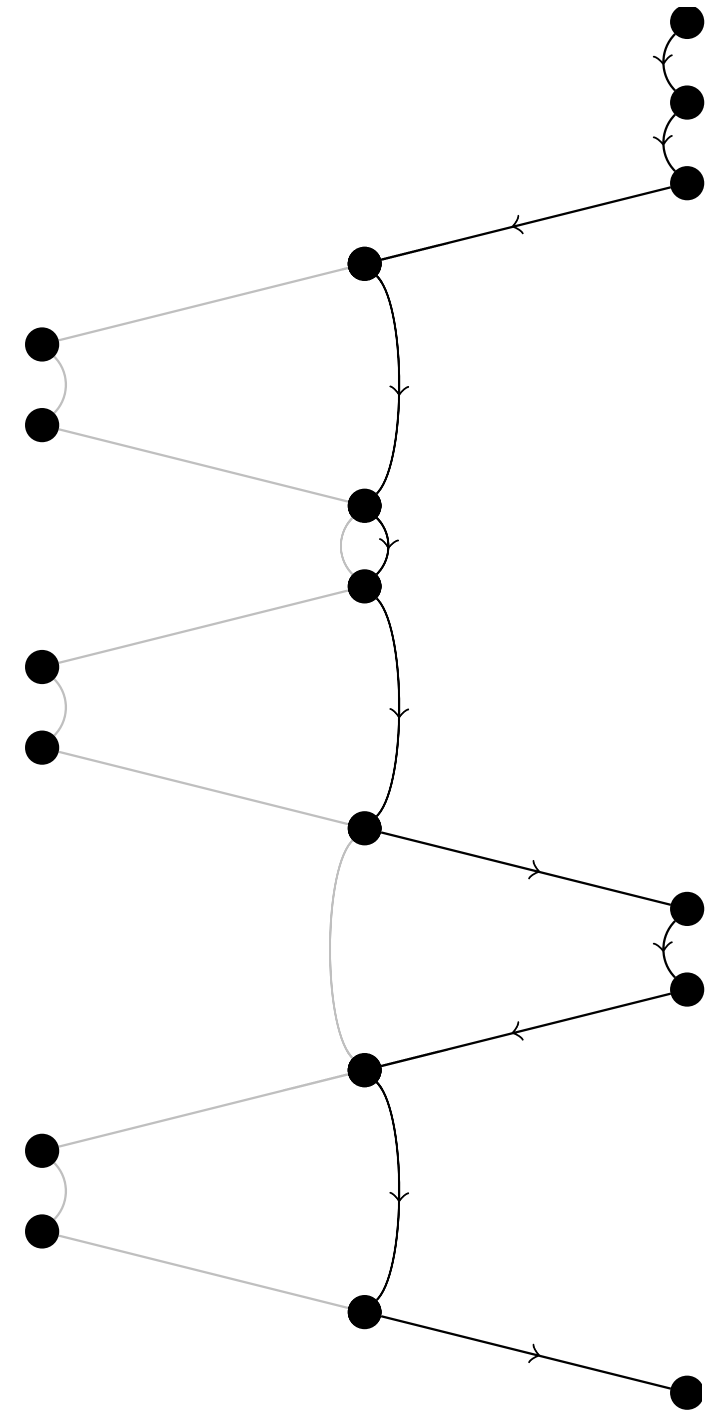

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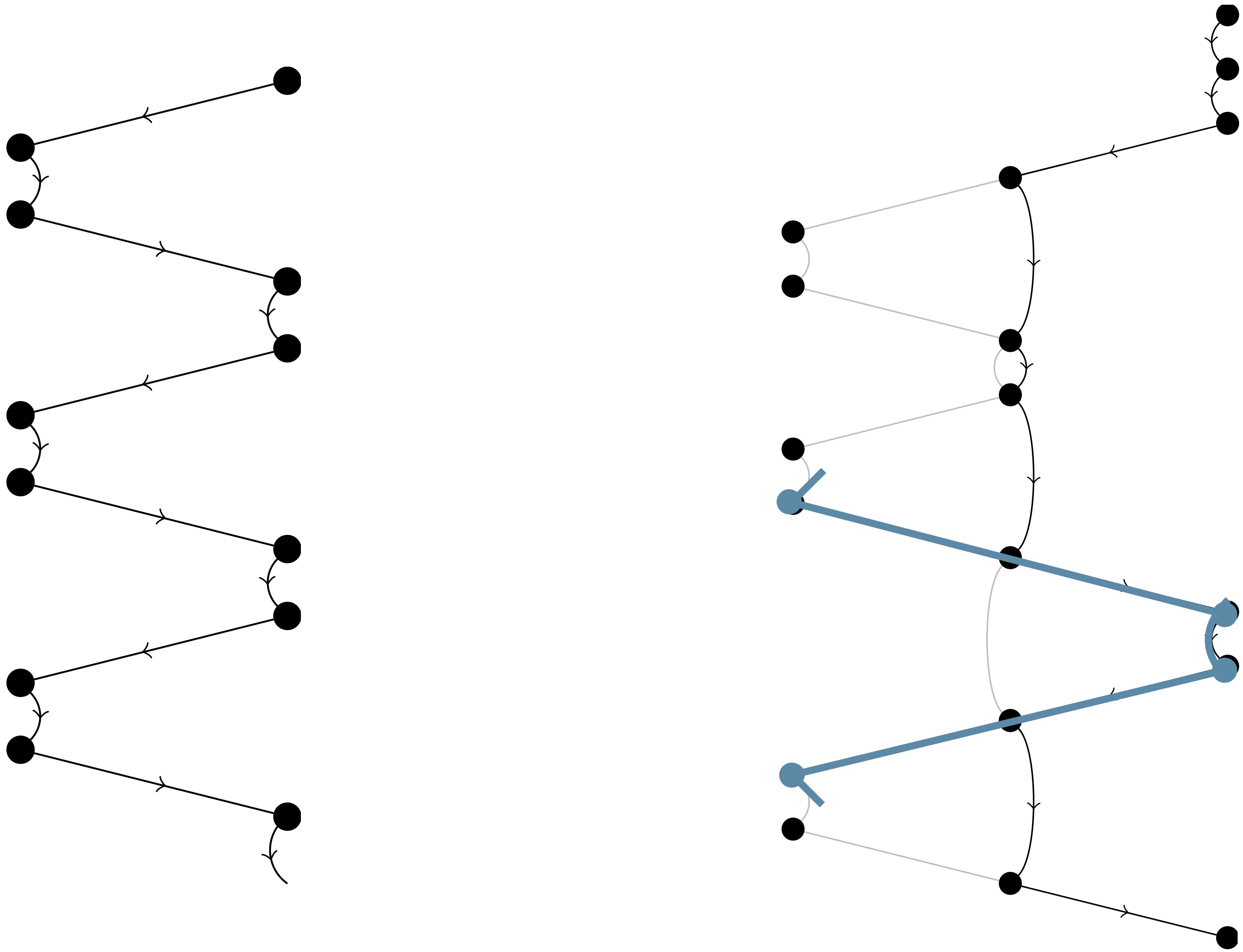

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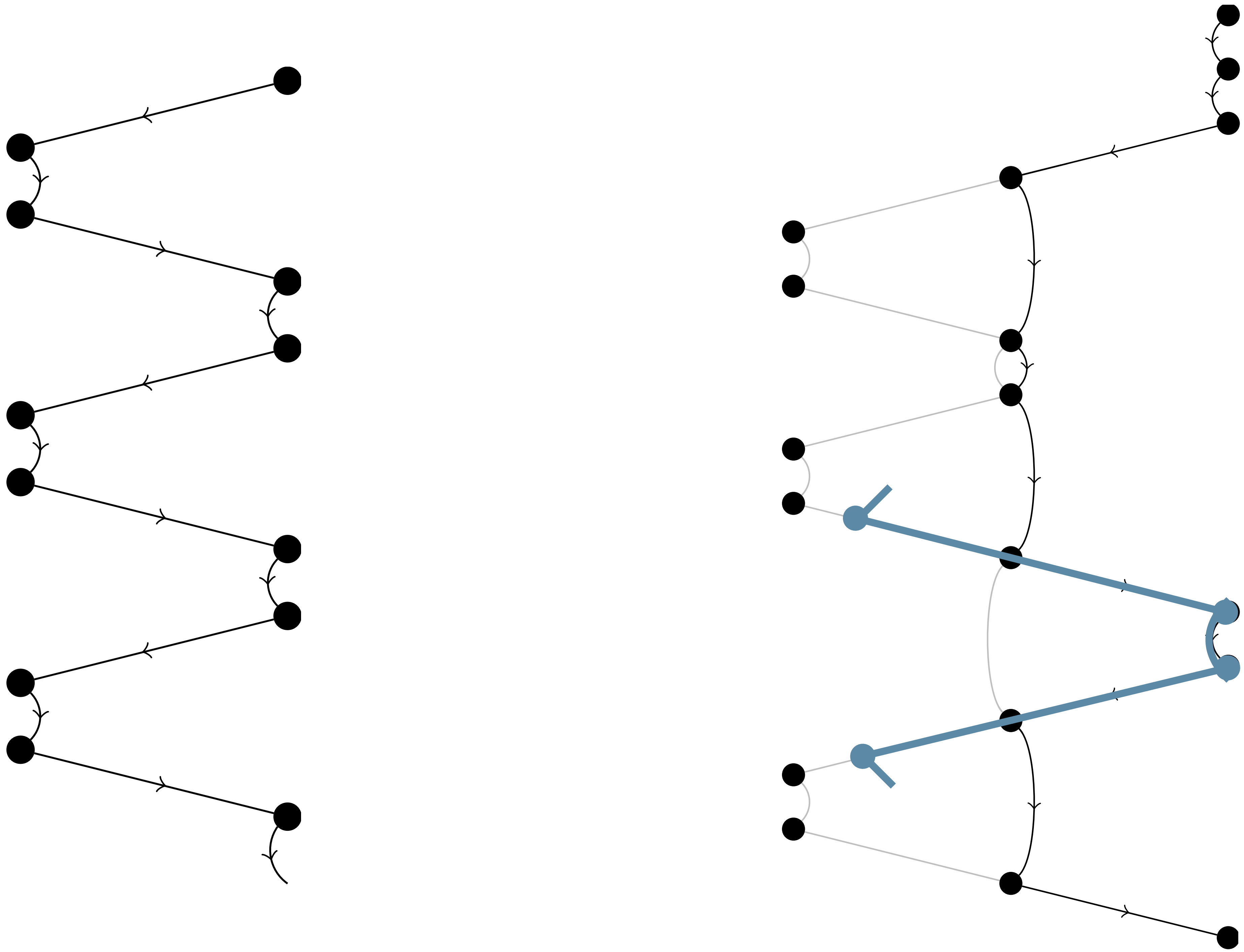

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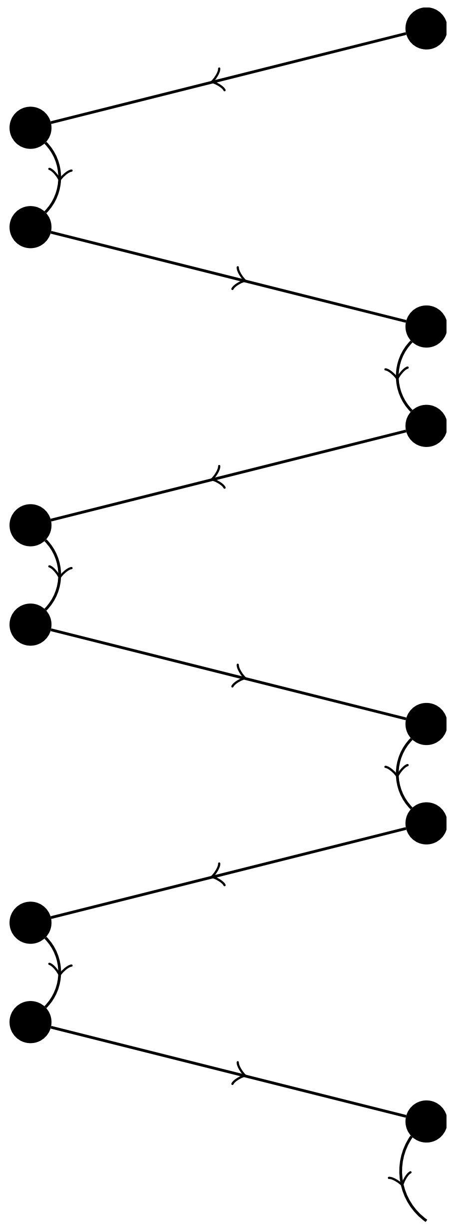

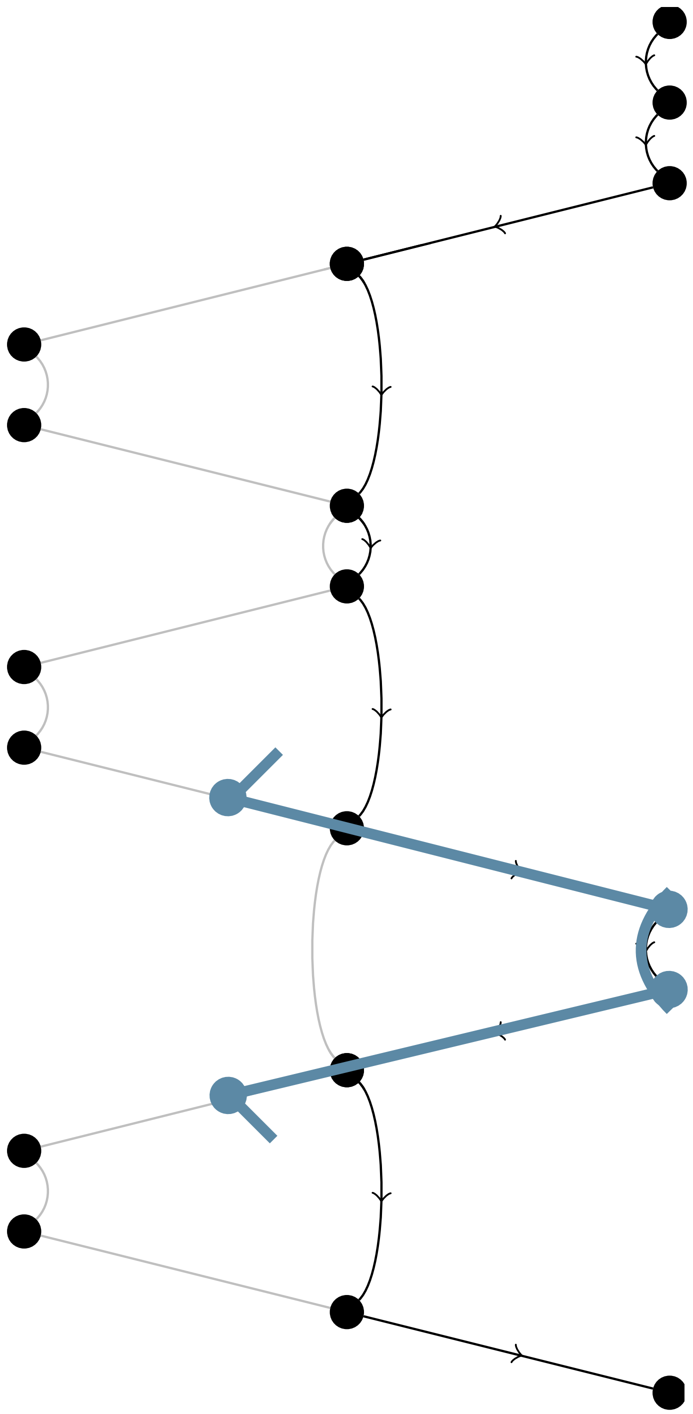

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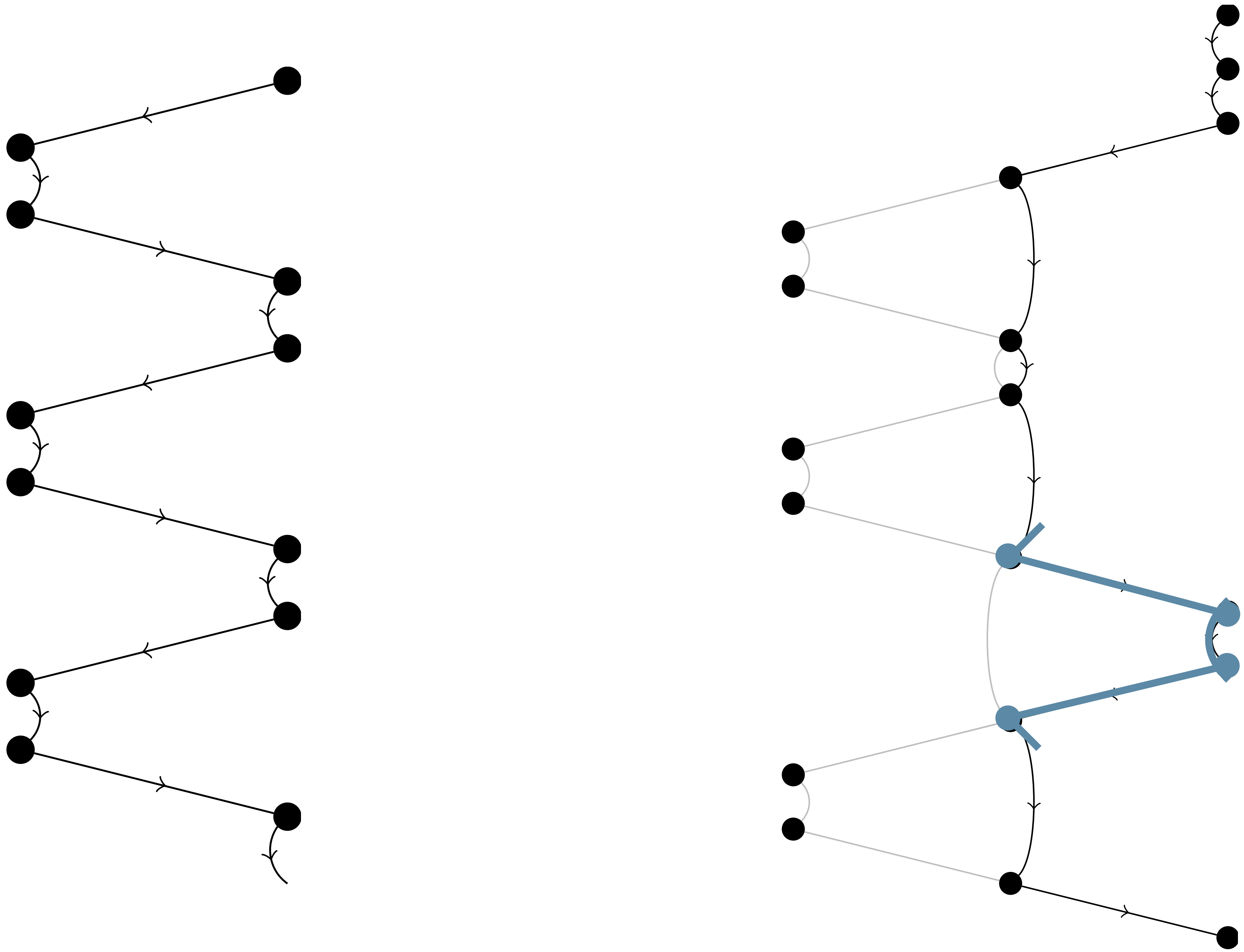

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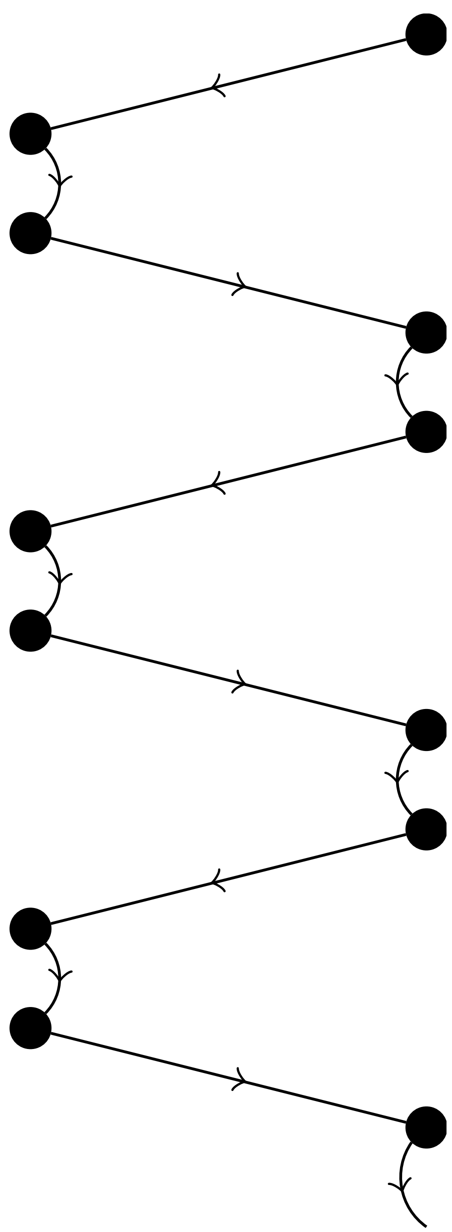

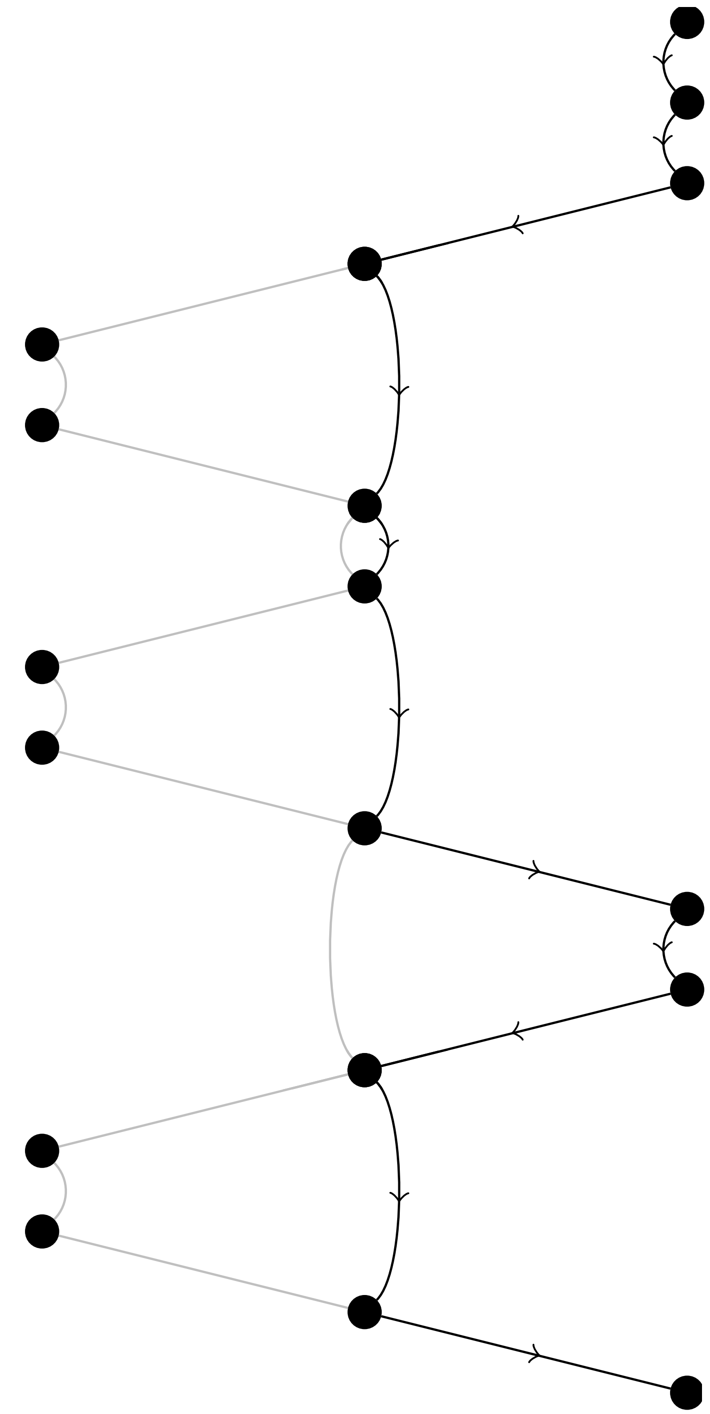

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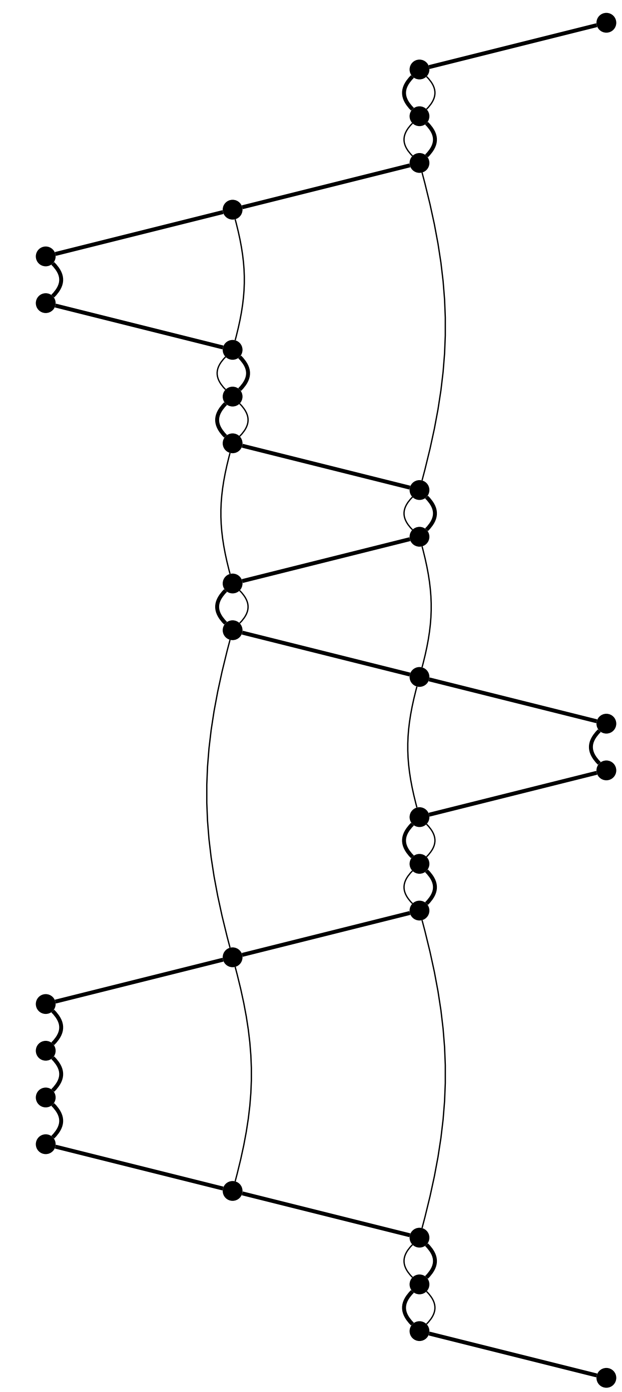

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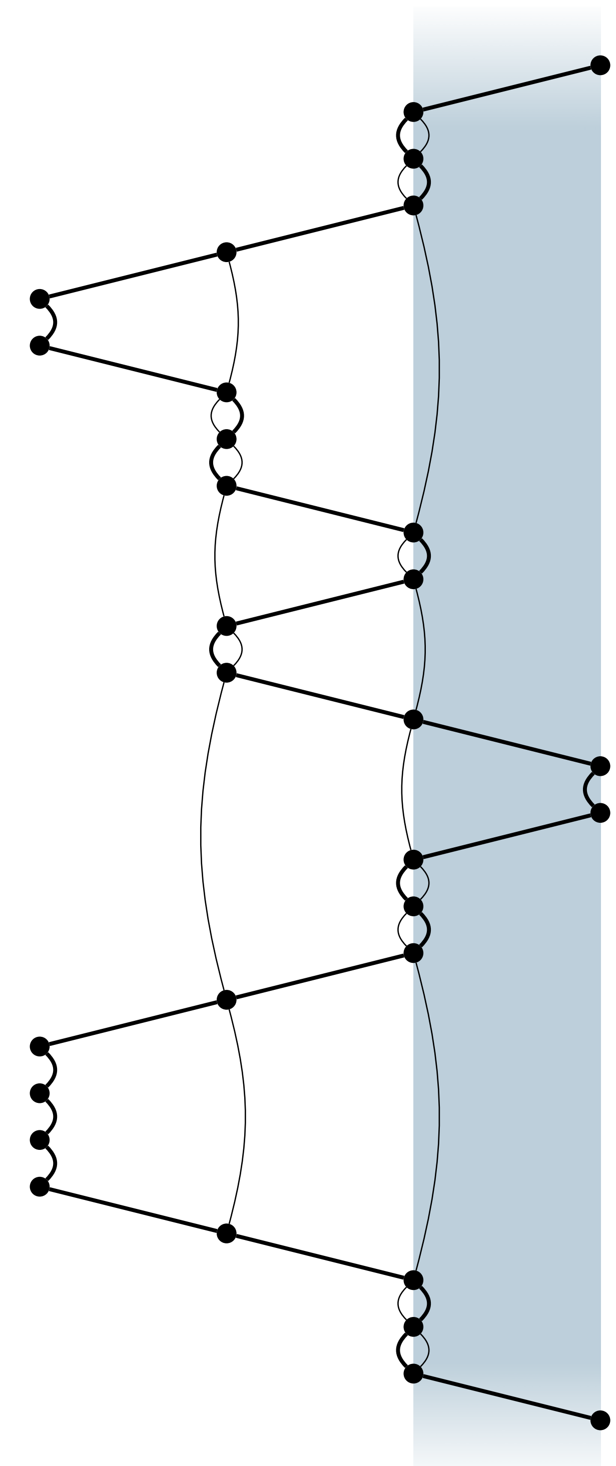

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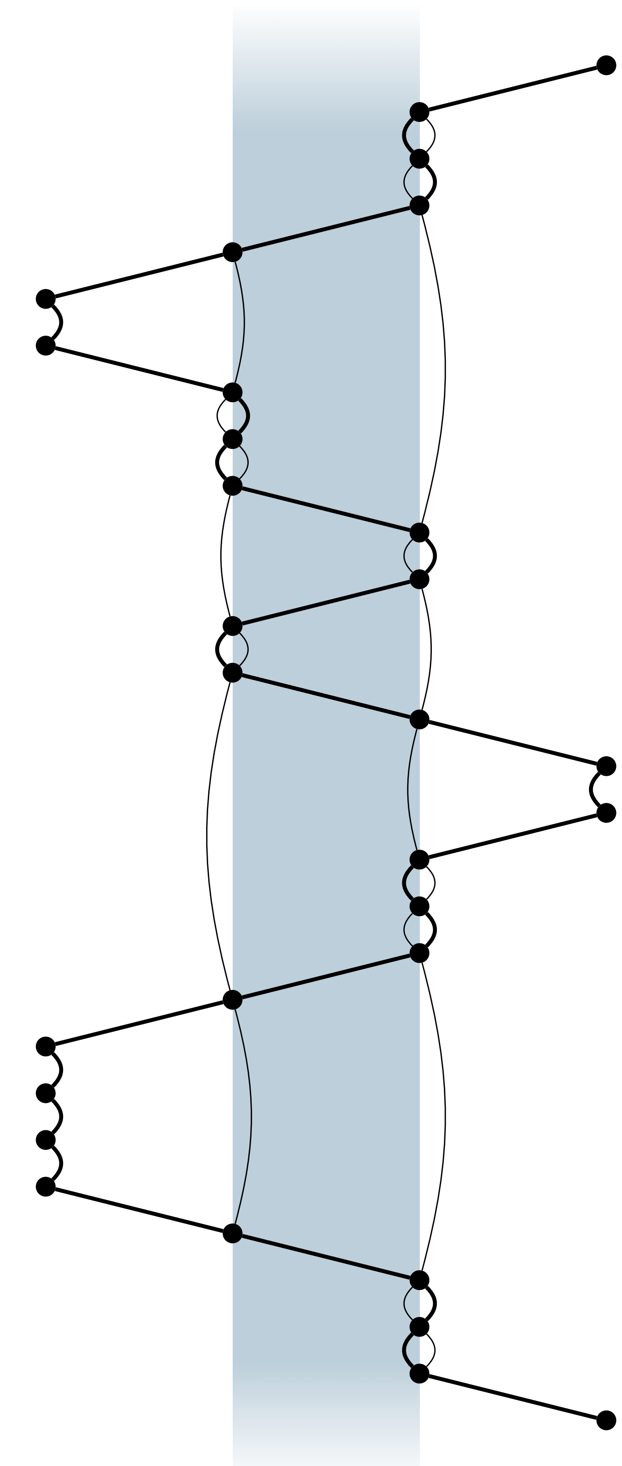

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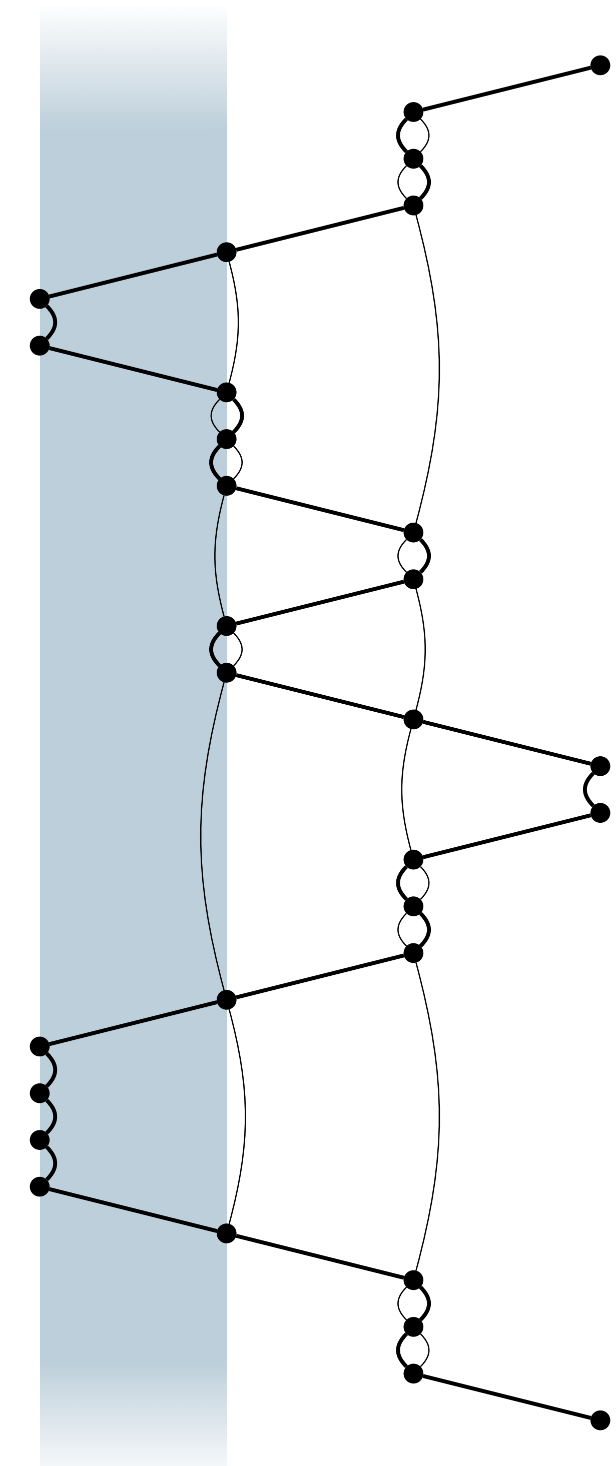

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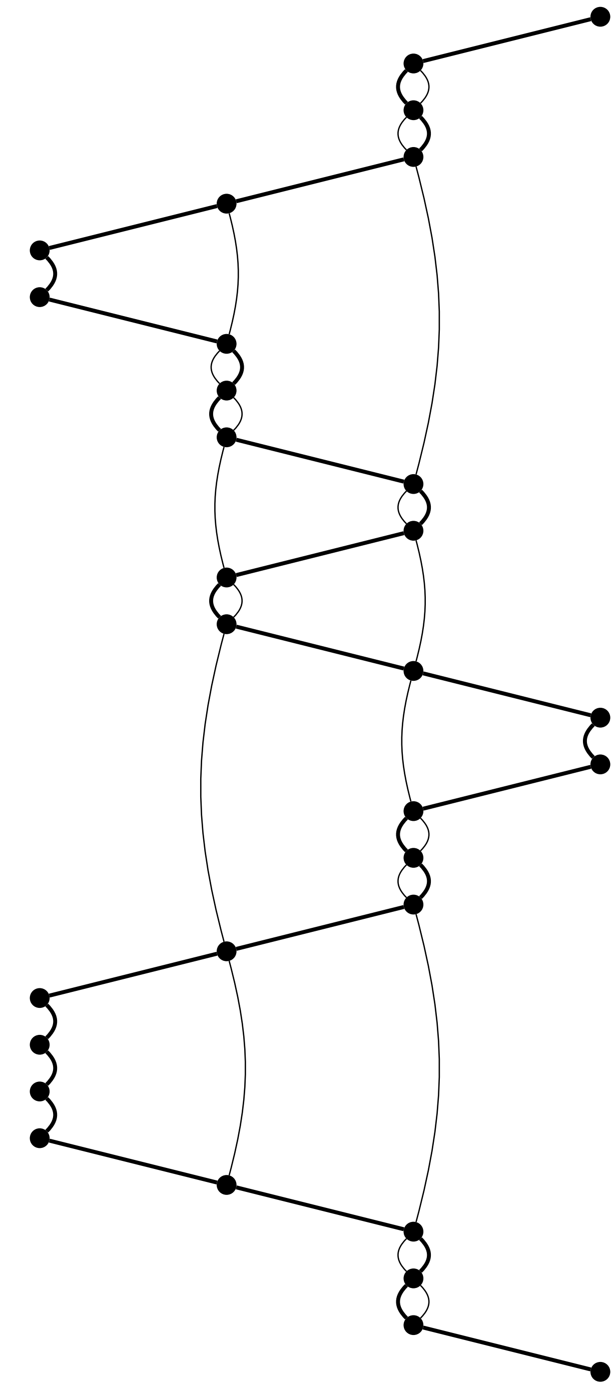

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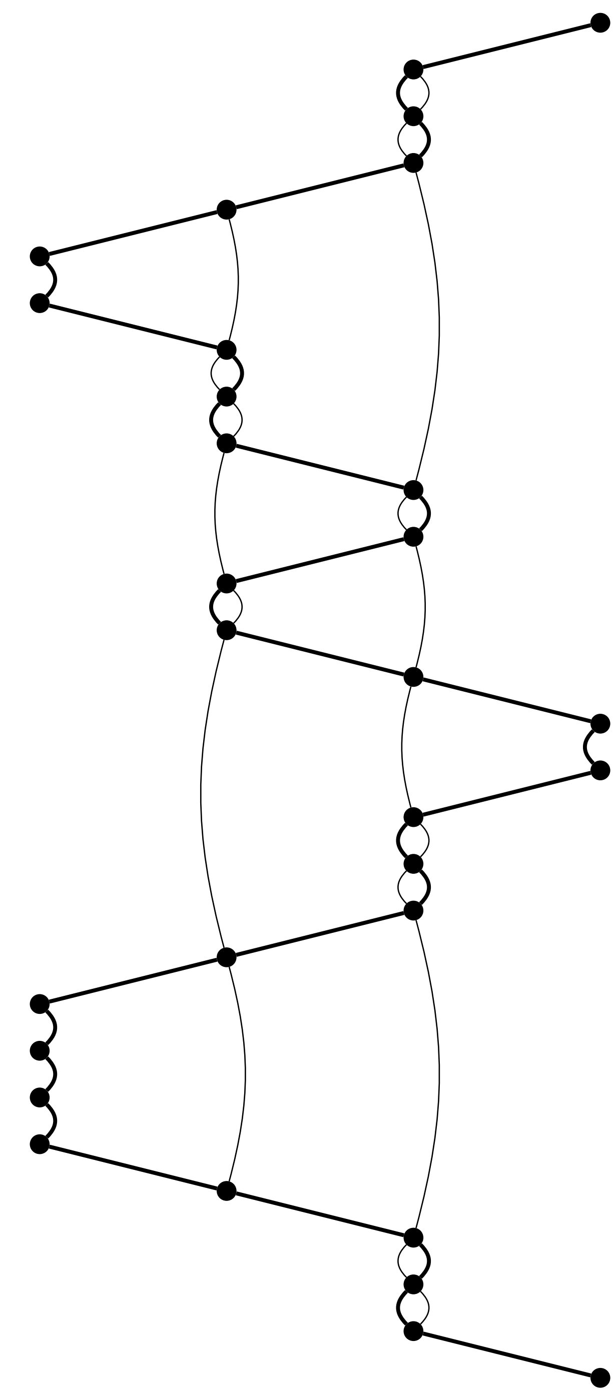

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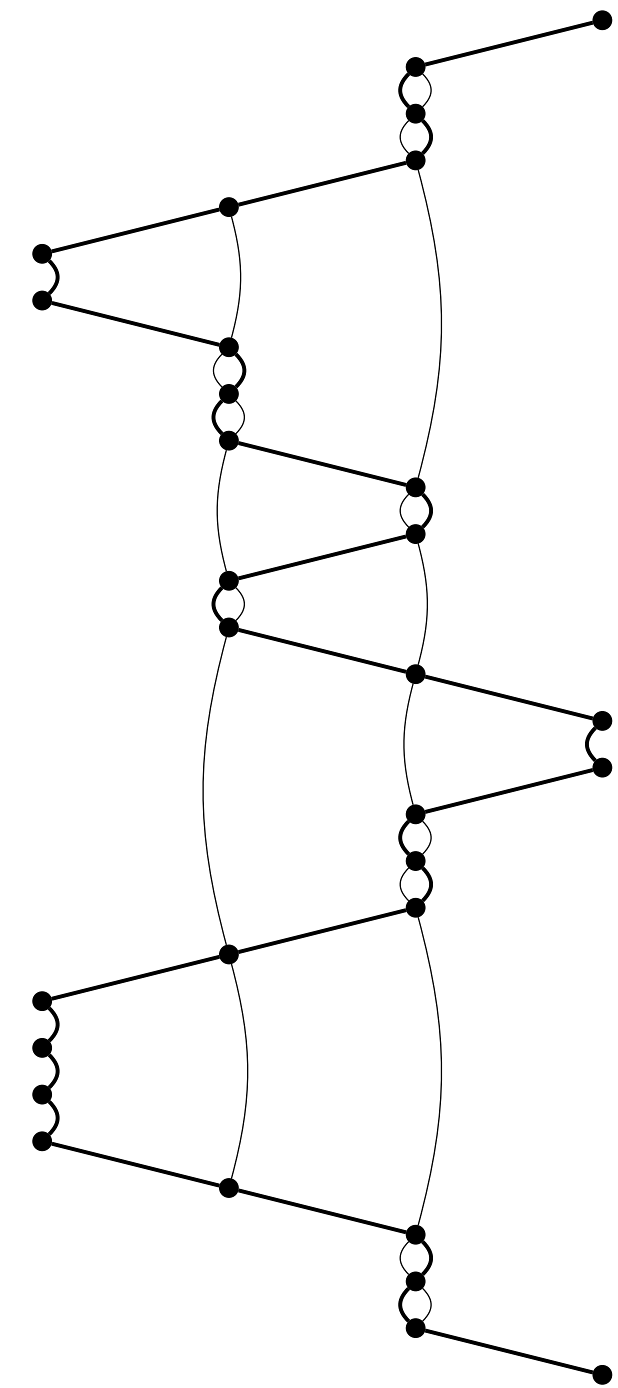

Associativity of composition

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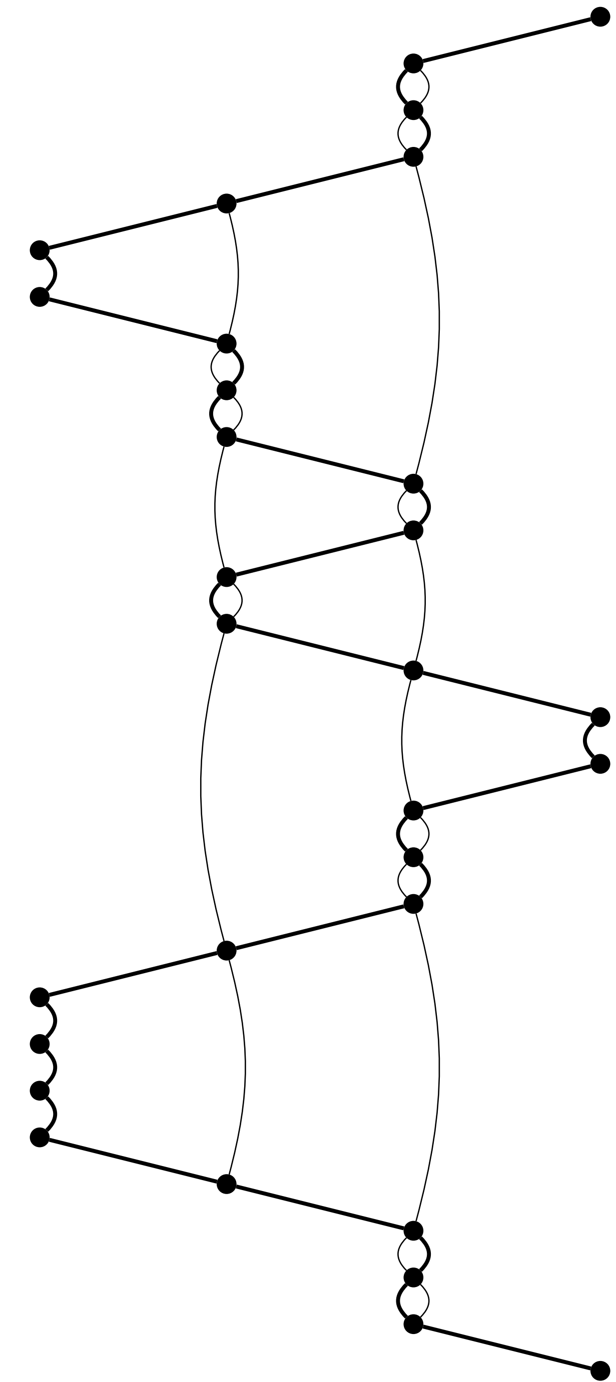
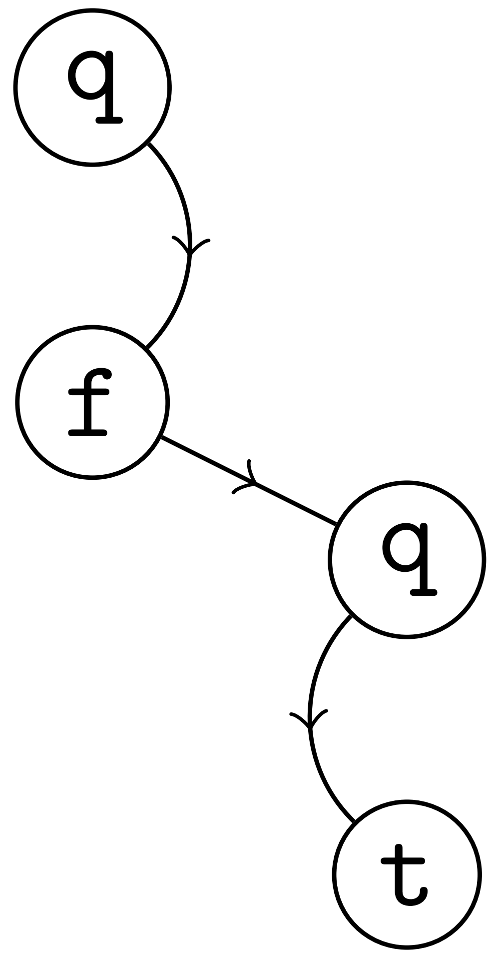
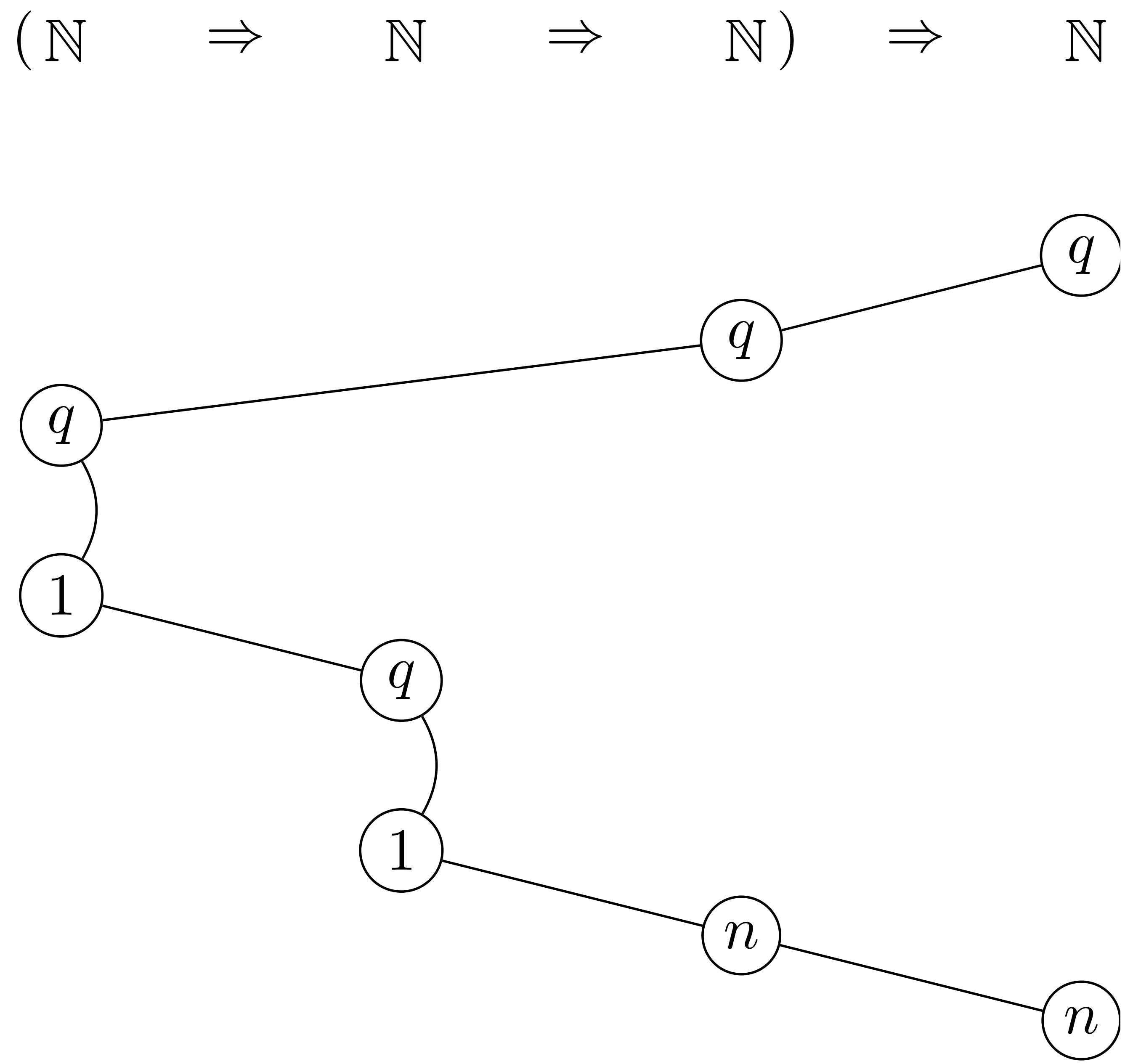
- Also shows that composition of strategies is associative, giving a category of graphical games.



Schedules for $\otimes$



Interleaving graphs



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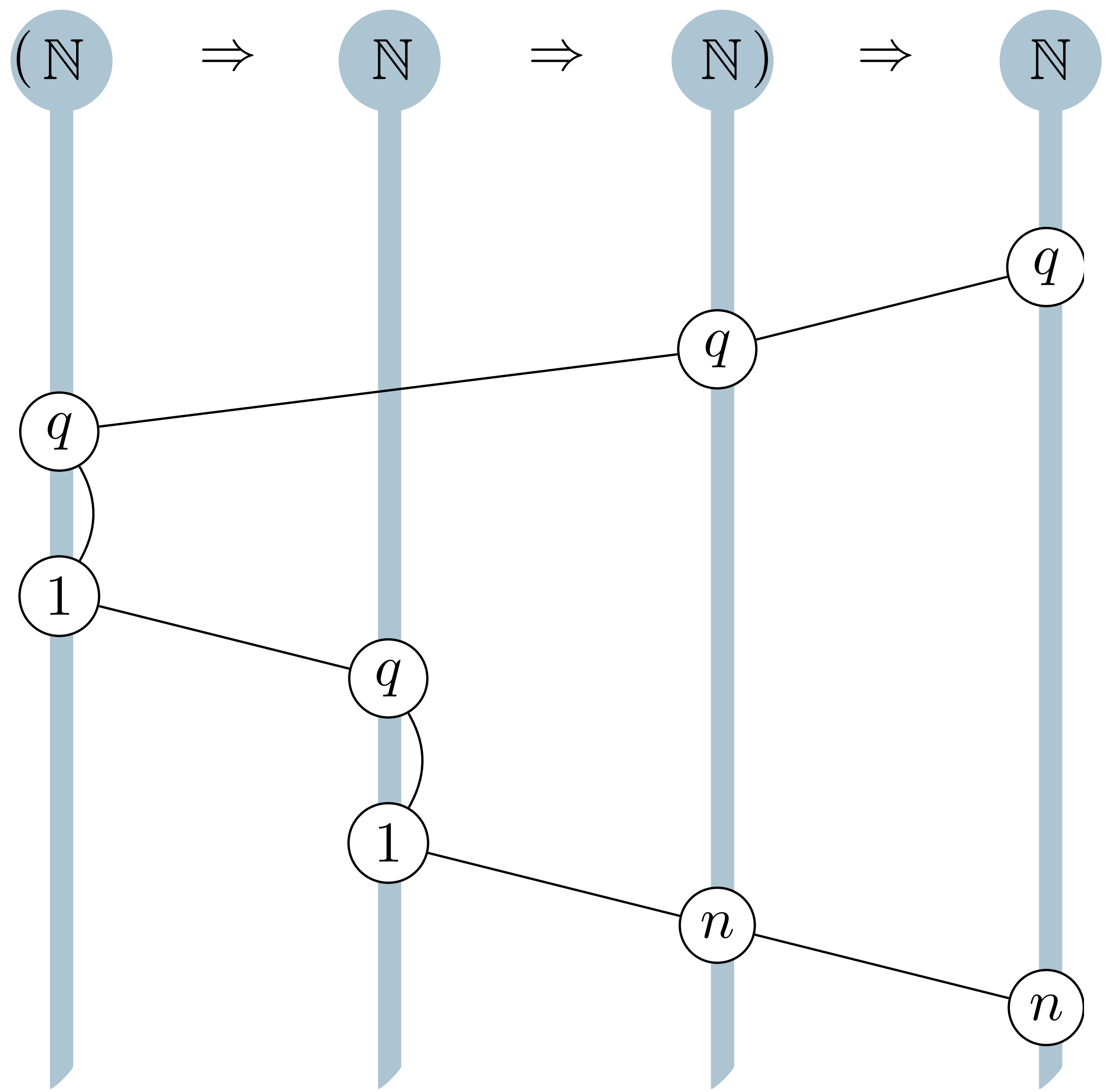
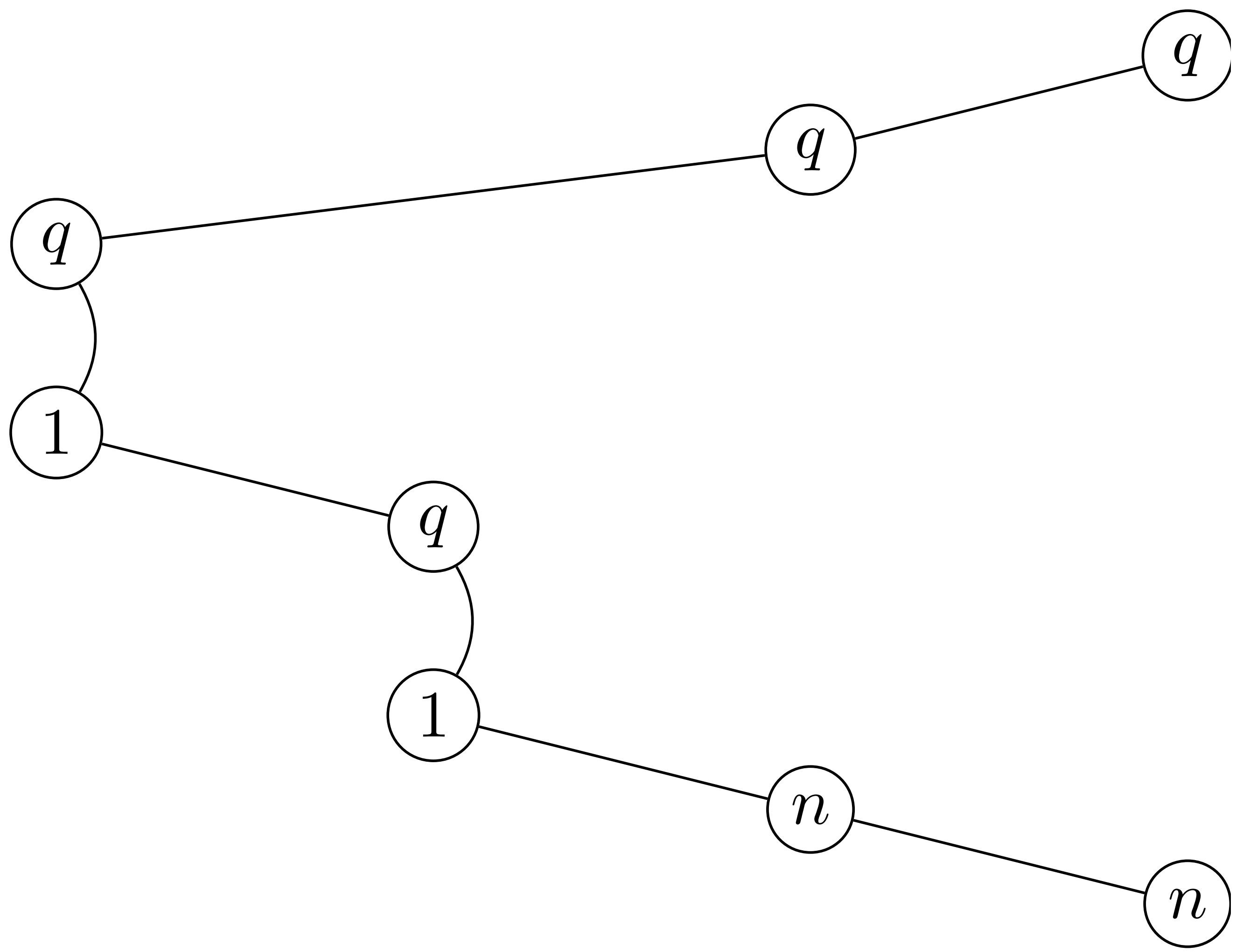
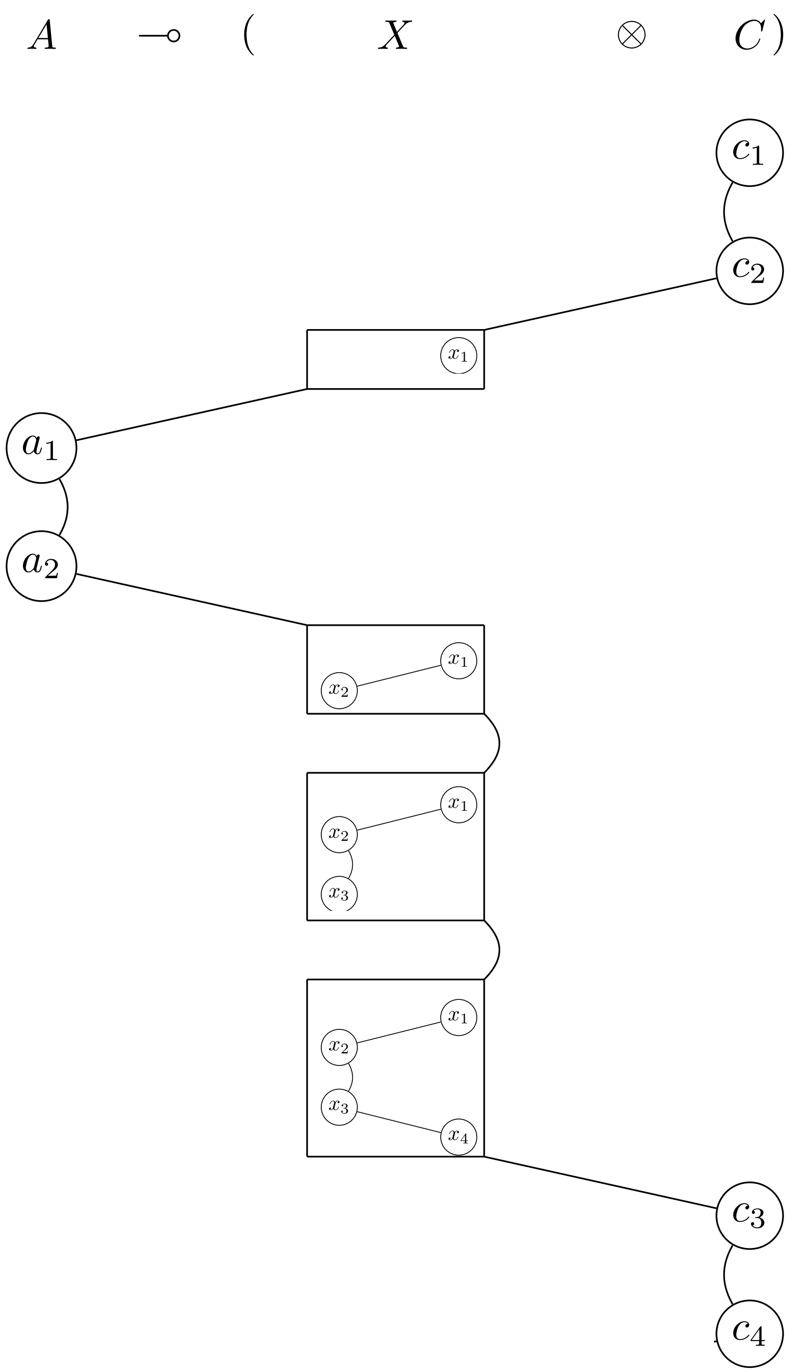


Interleaving graphs

$$(\mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathbb{N}) \Rightarrow \mathbb{N}$$



Two representations of plays: unfolding and folding



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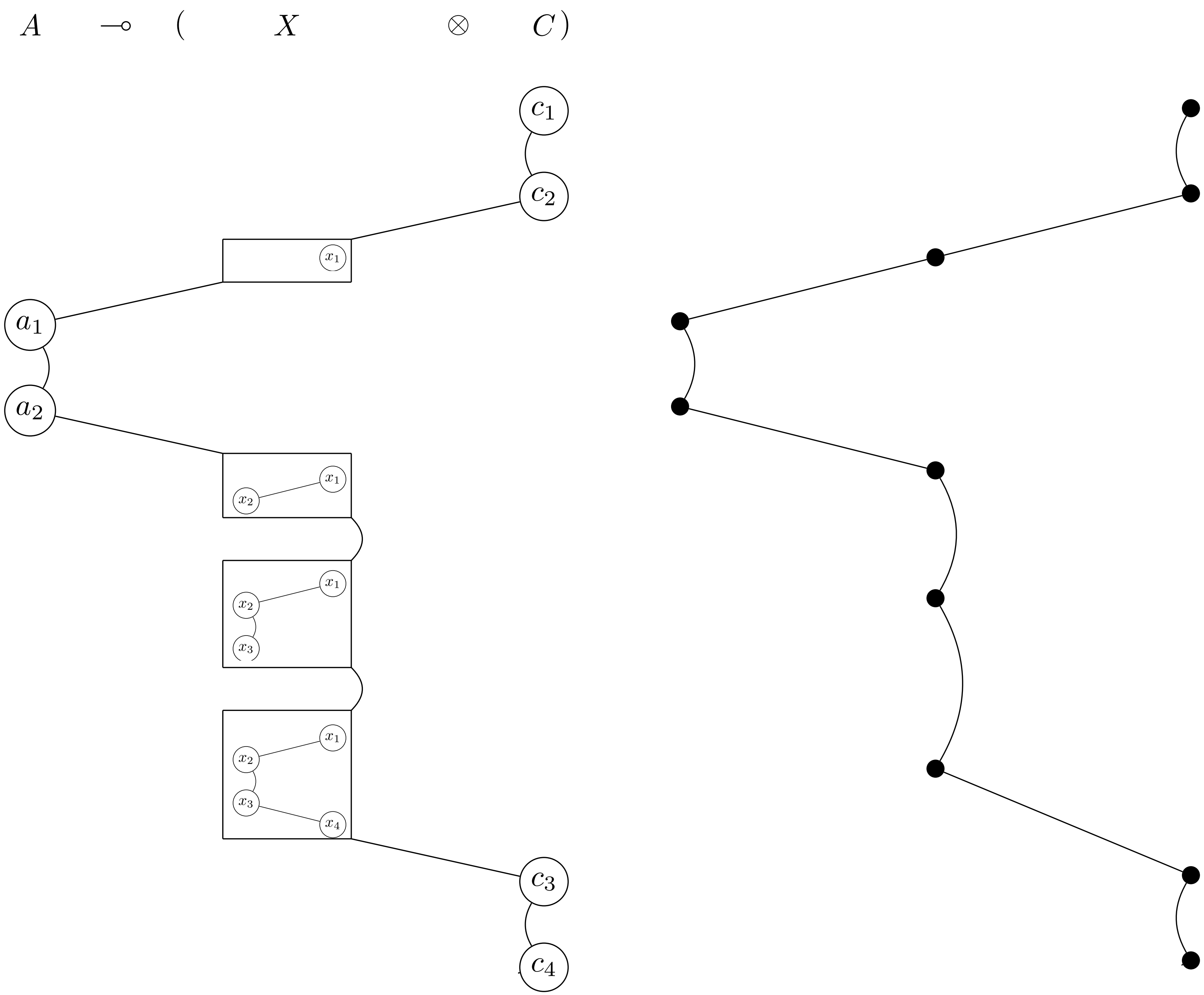

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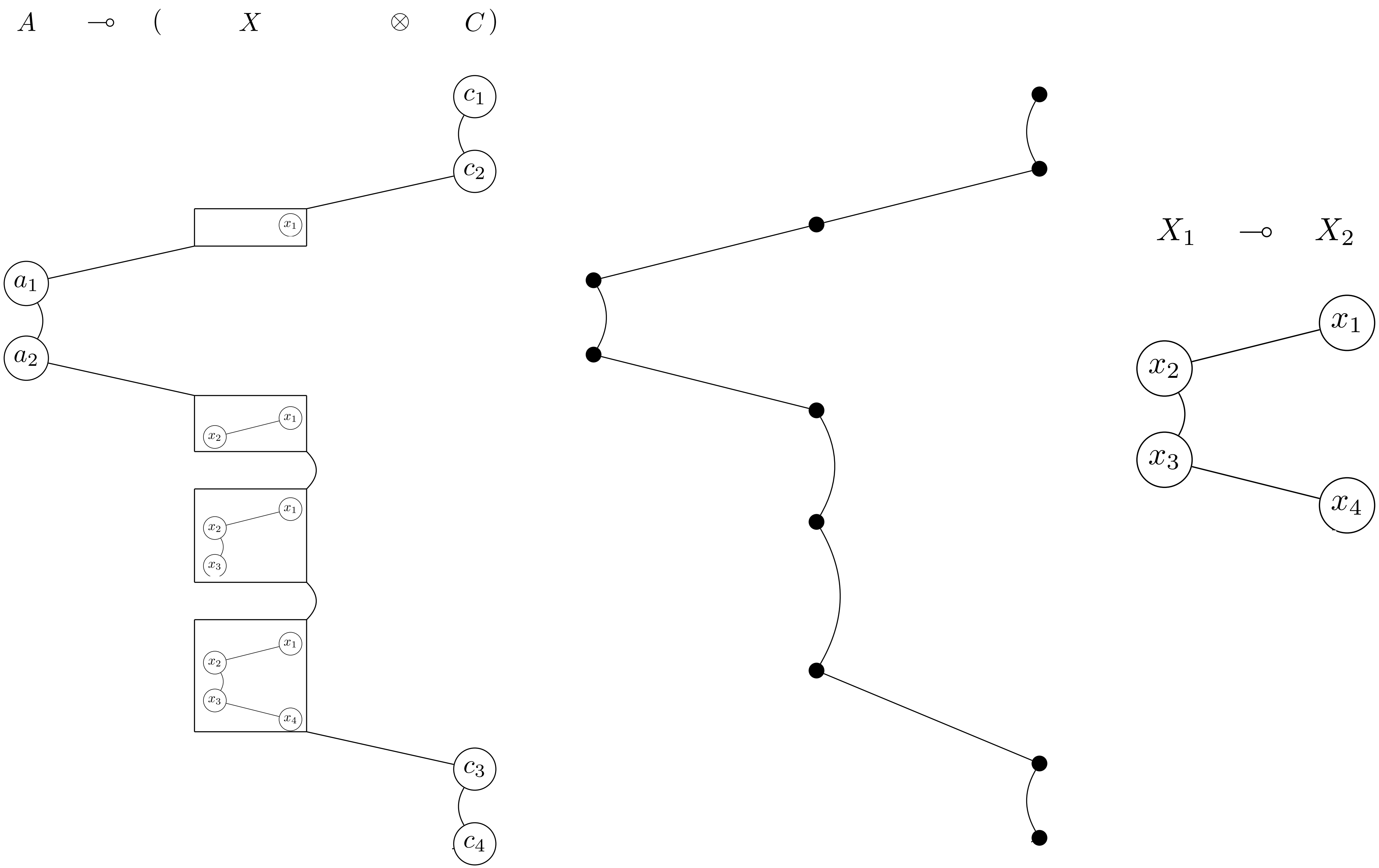

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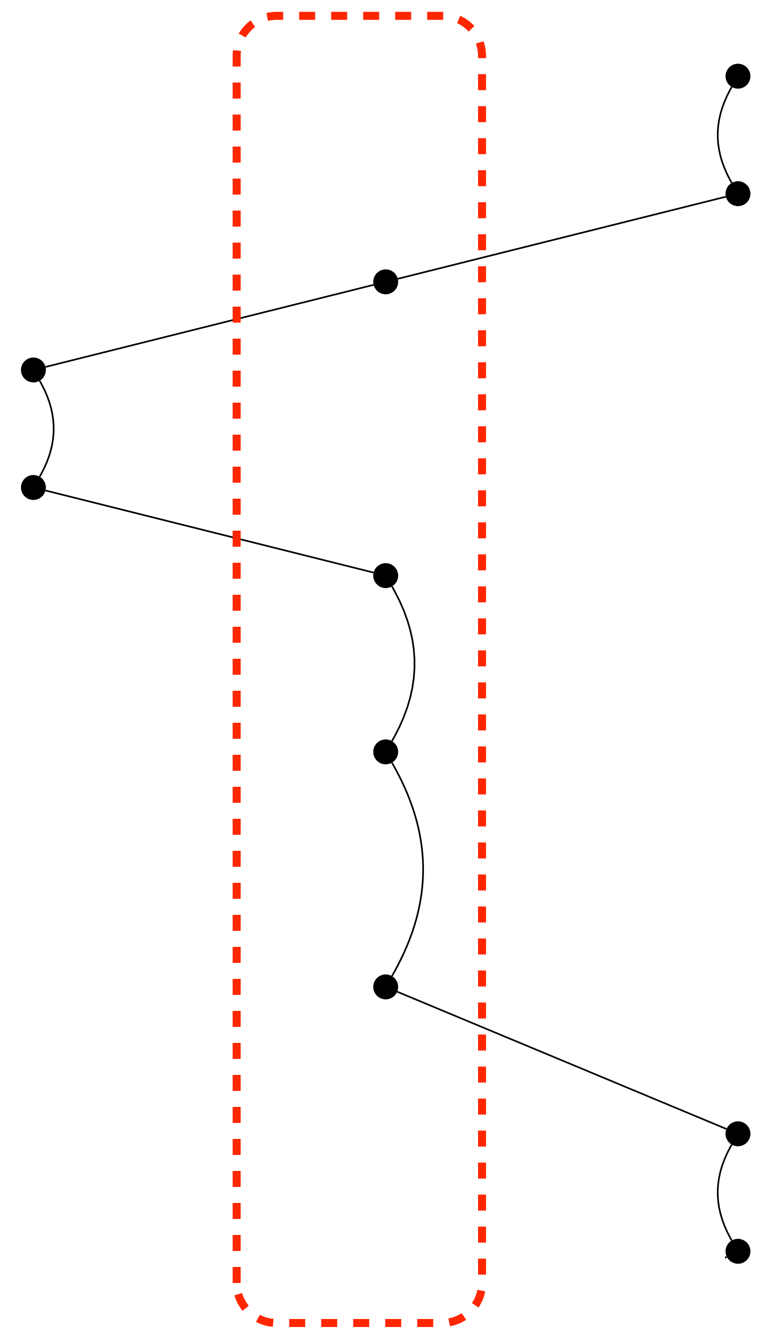

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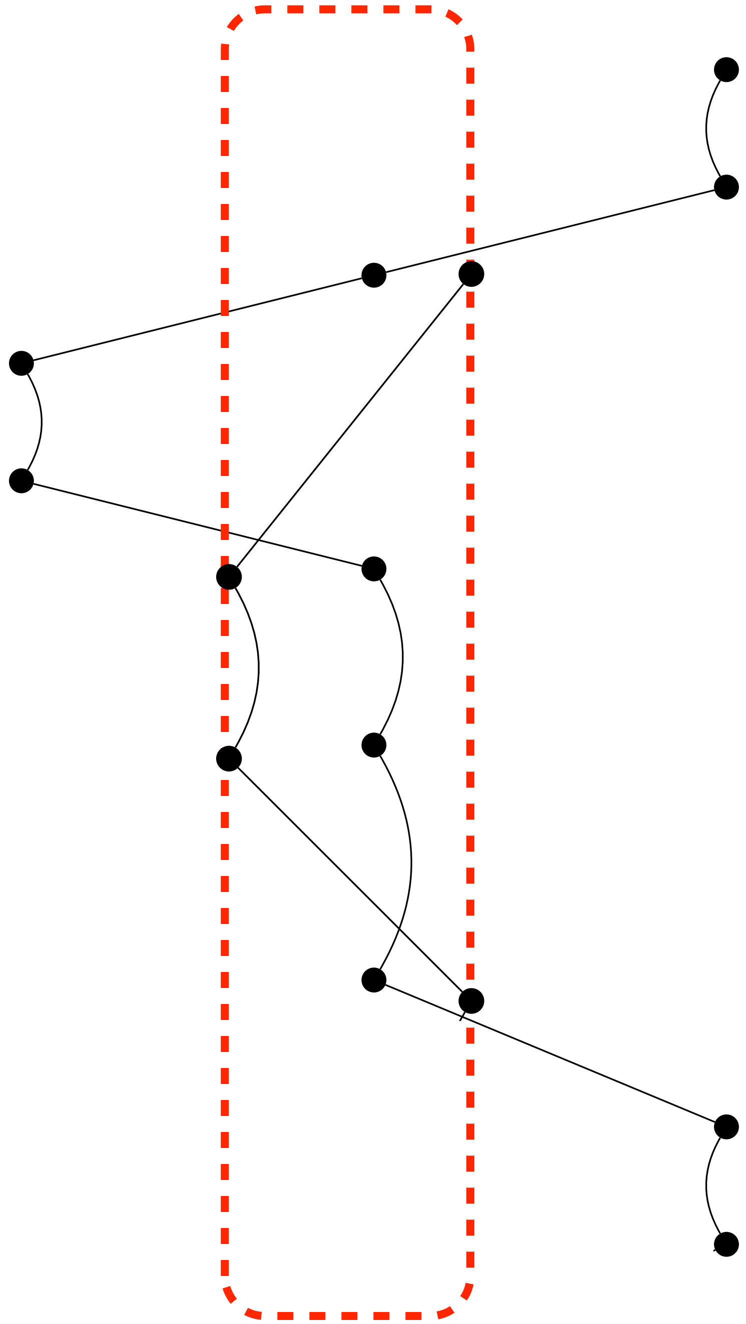

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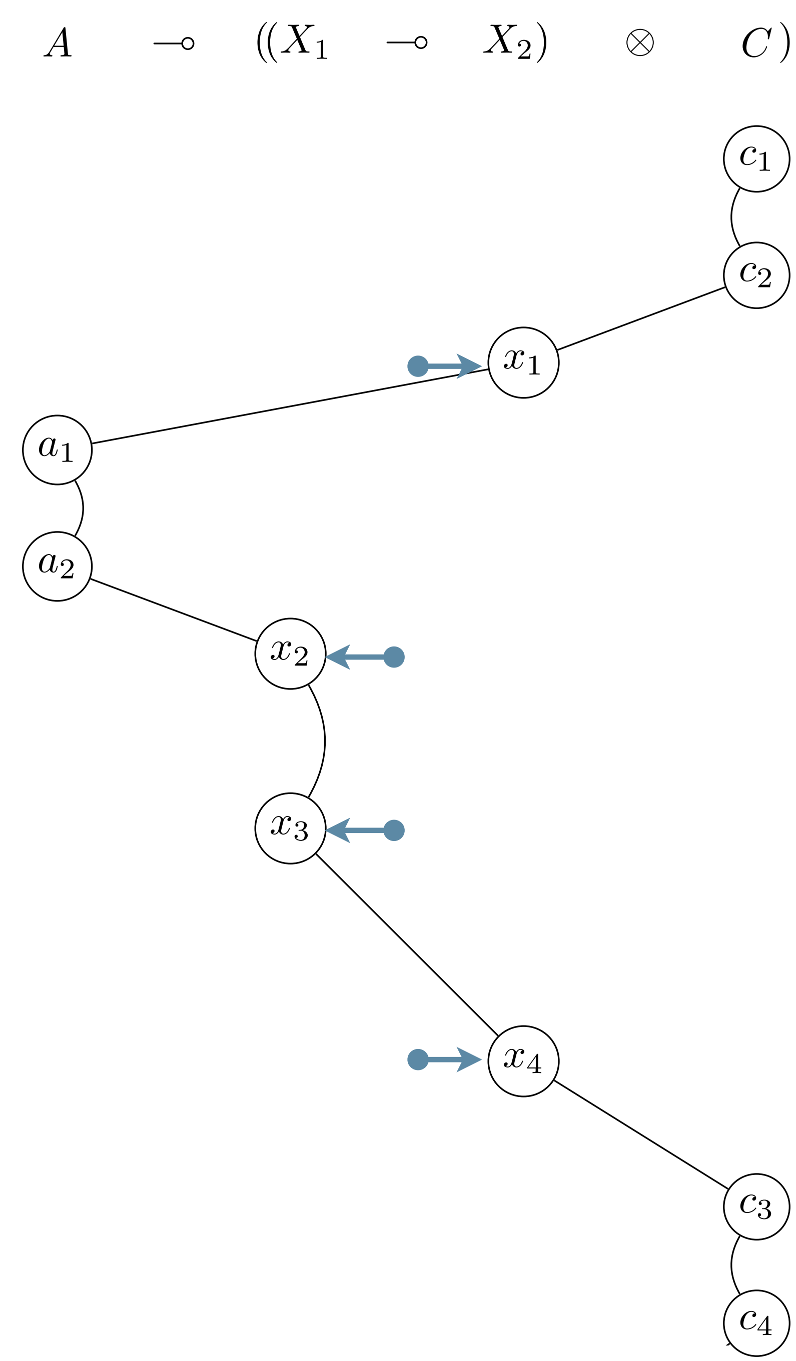

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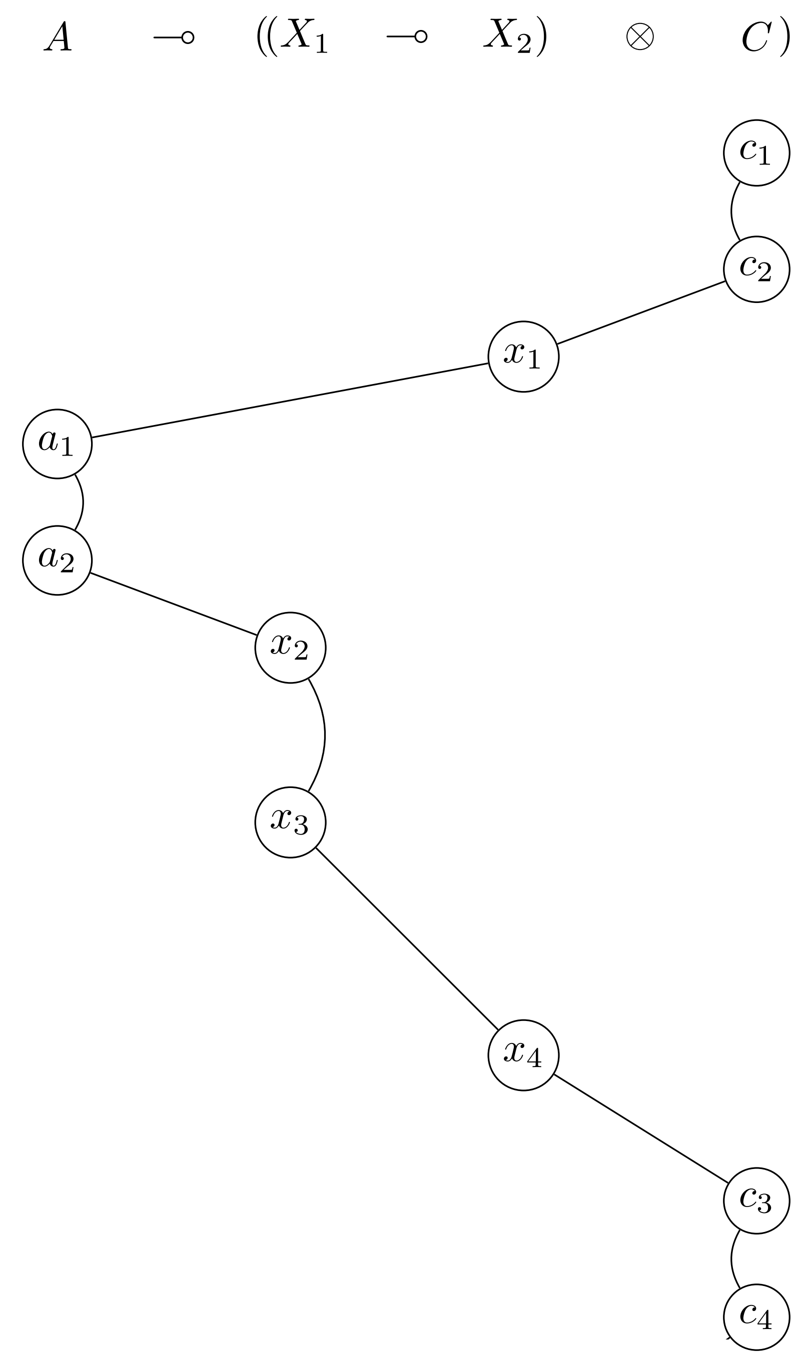

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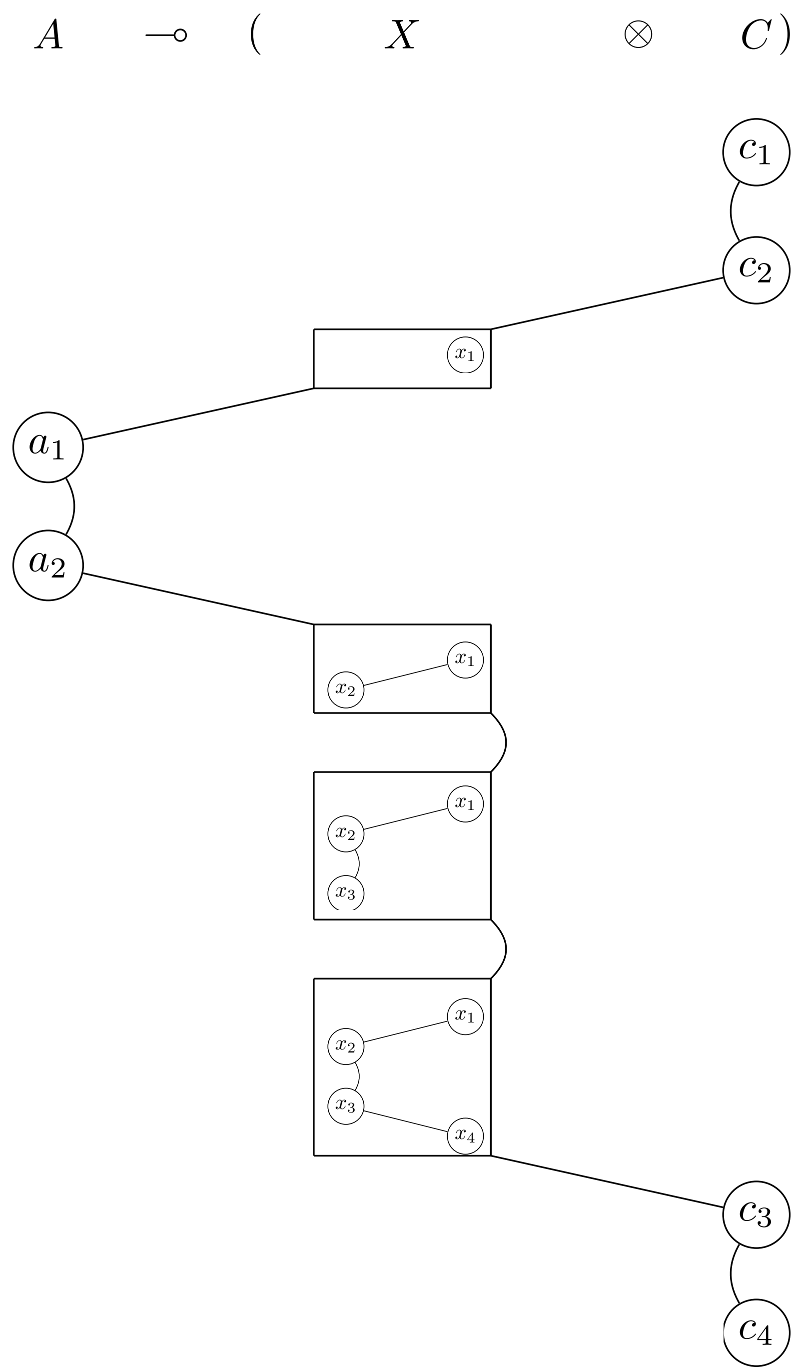

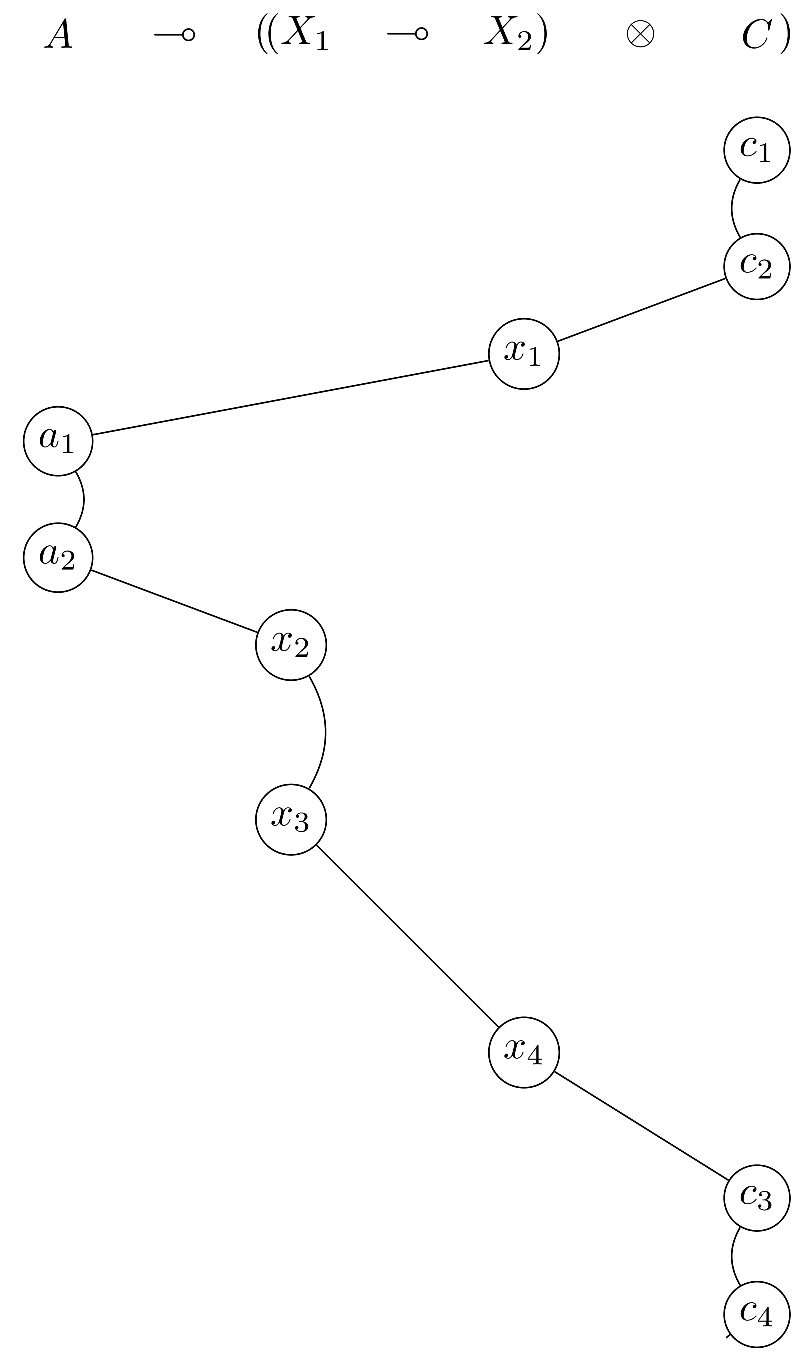

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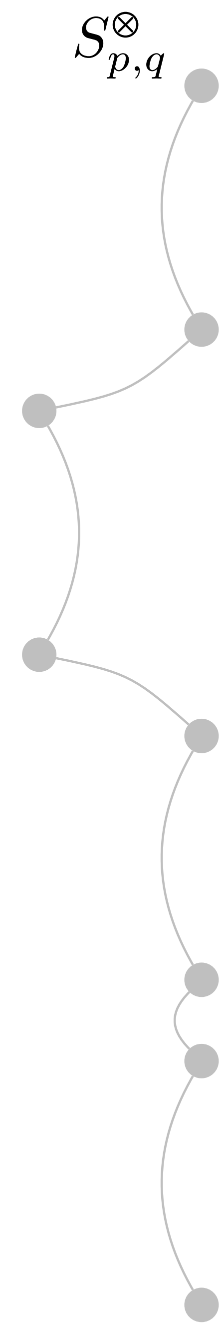
- We can say, for example:

  - "Every position of $(A \otimes B) \multimap C$ *is* a position of $A \multimap (B \multimap C)$. The first move is in C, subsequent moves come in pairs in A, B or C."

Symmetry



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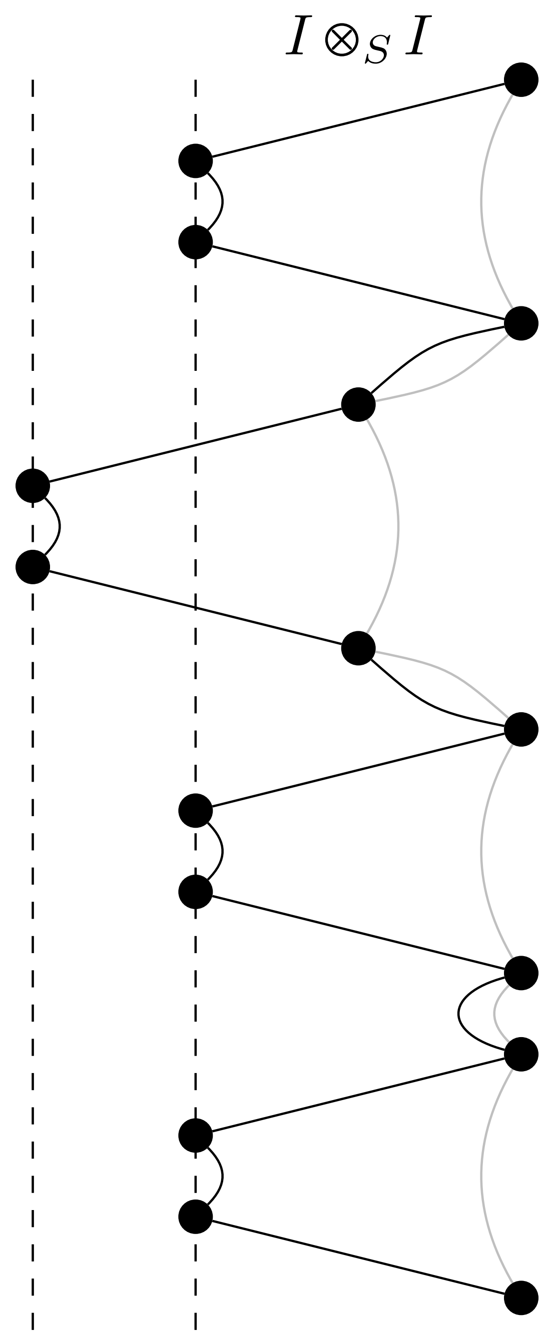

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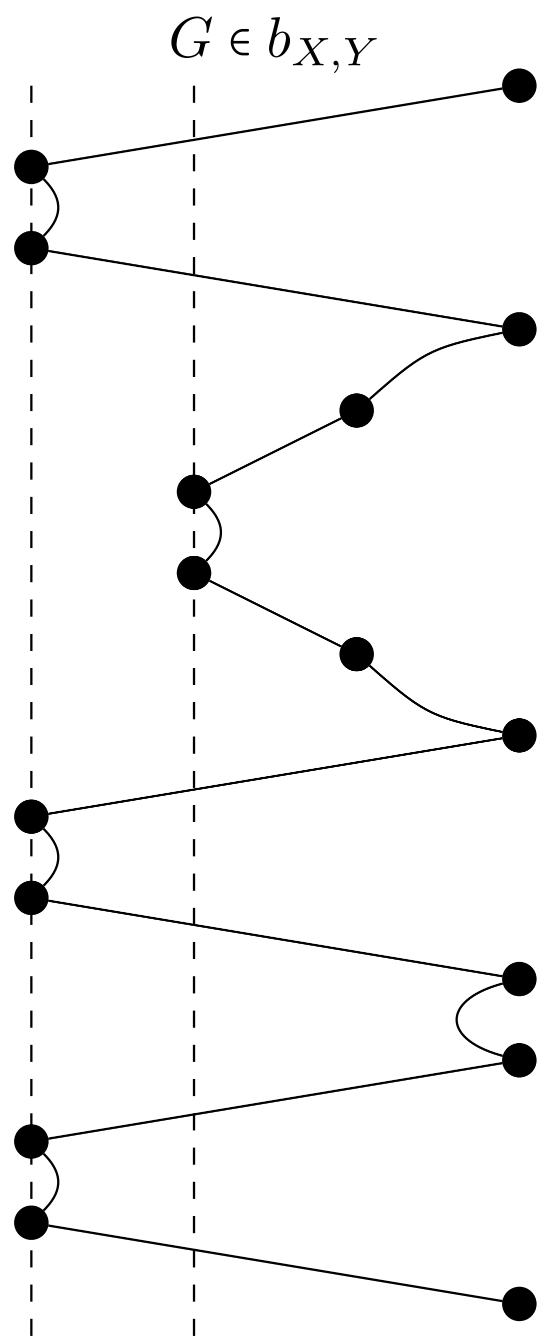

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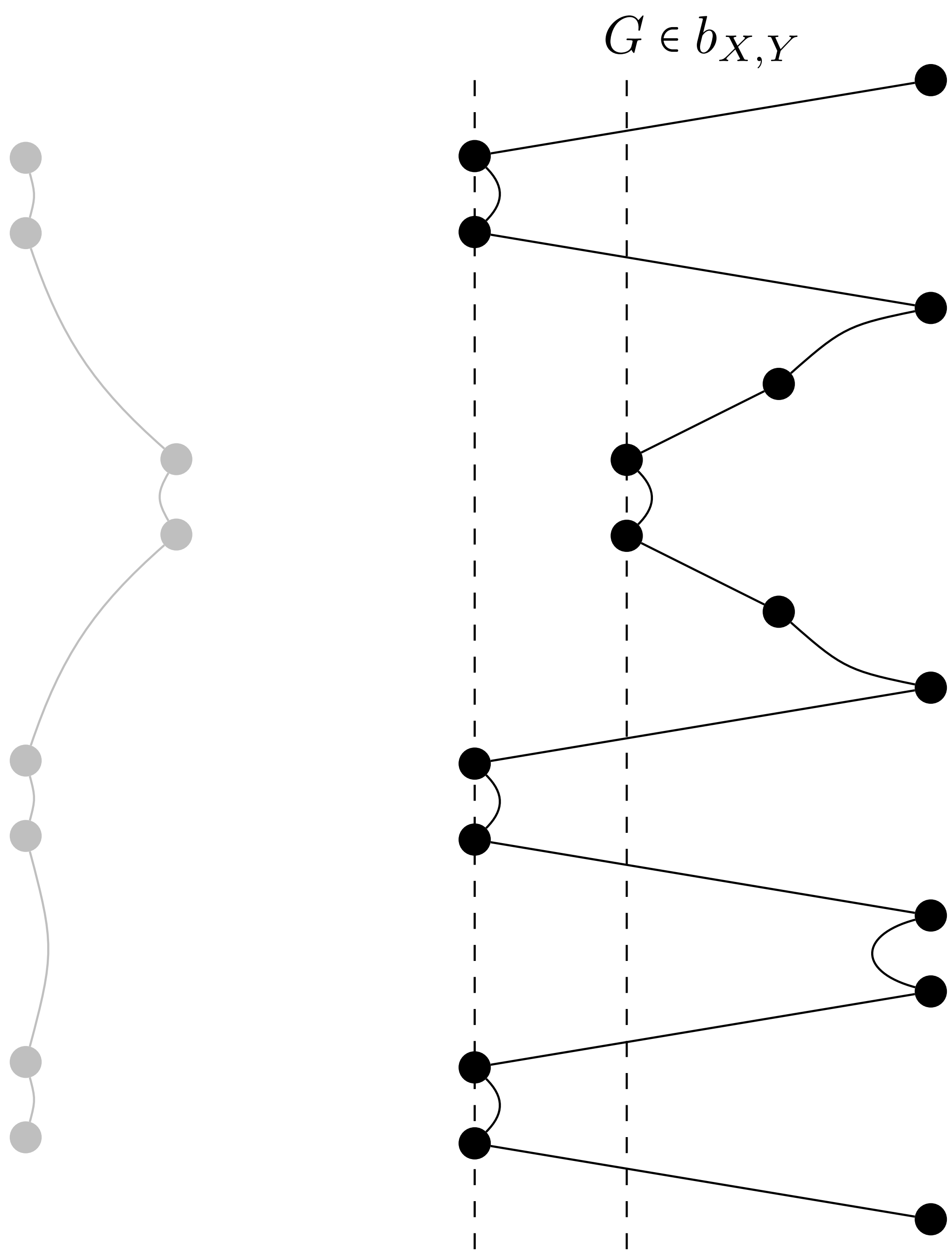

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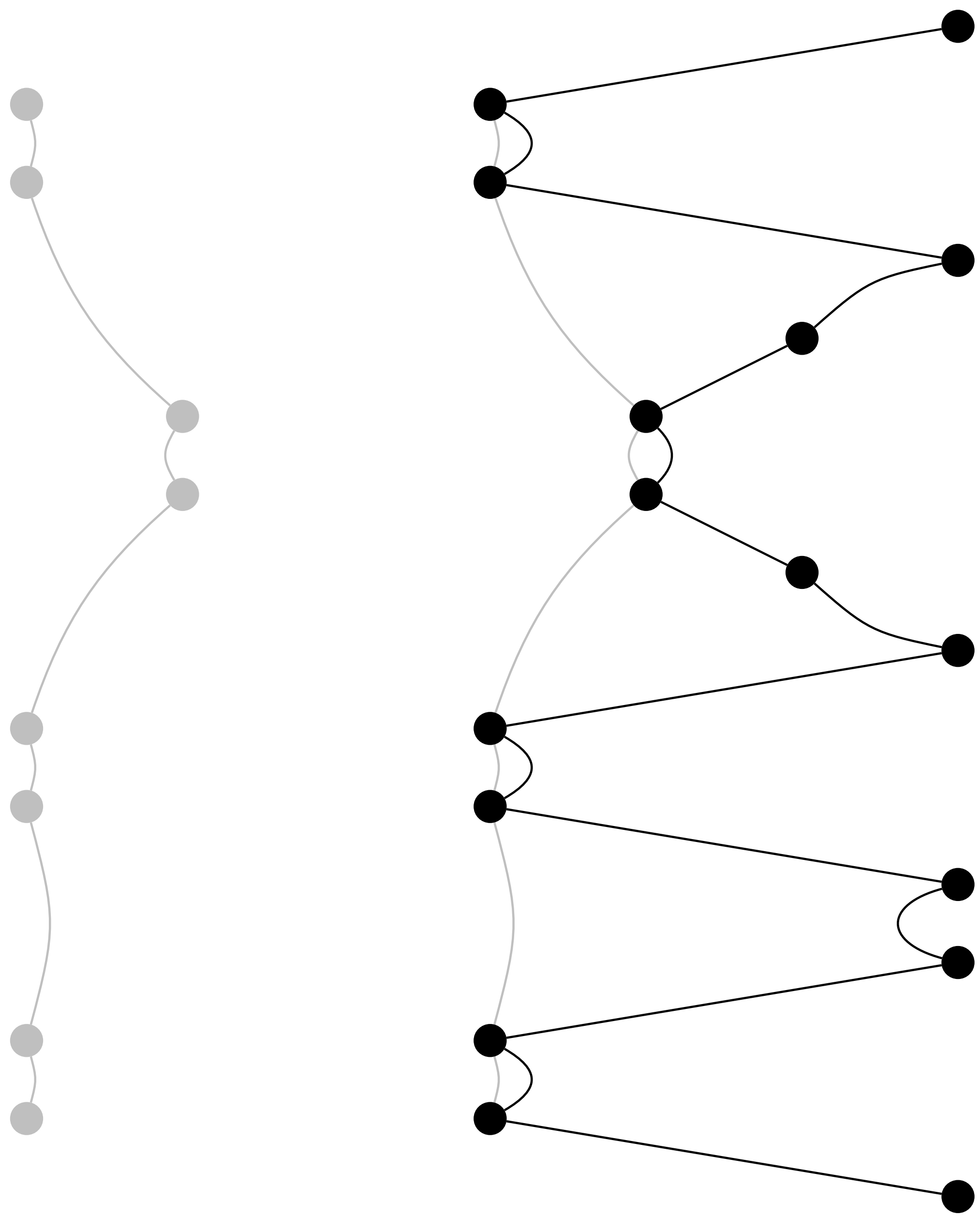

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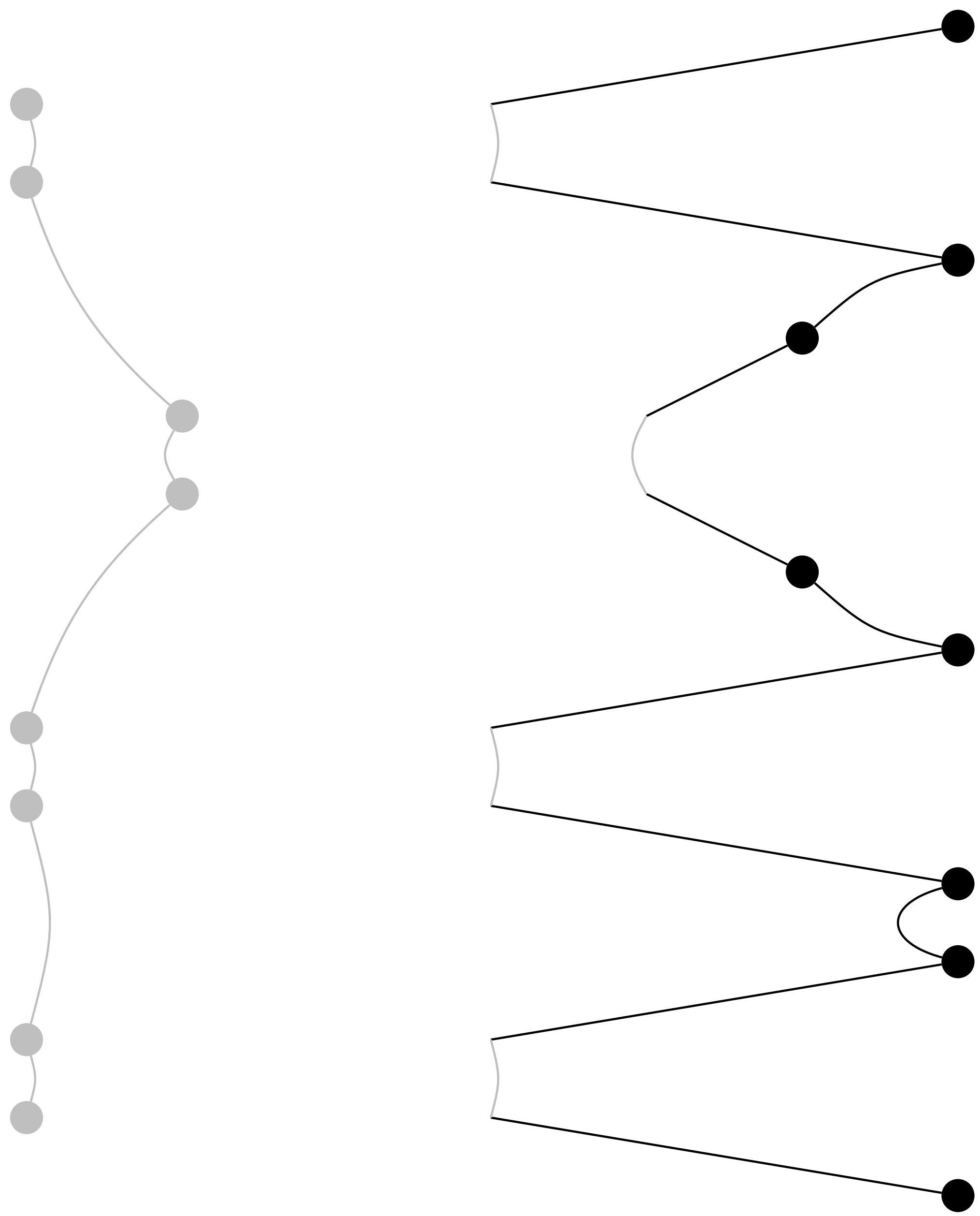

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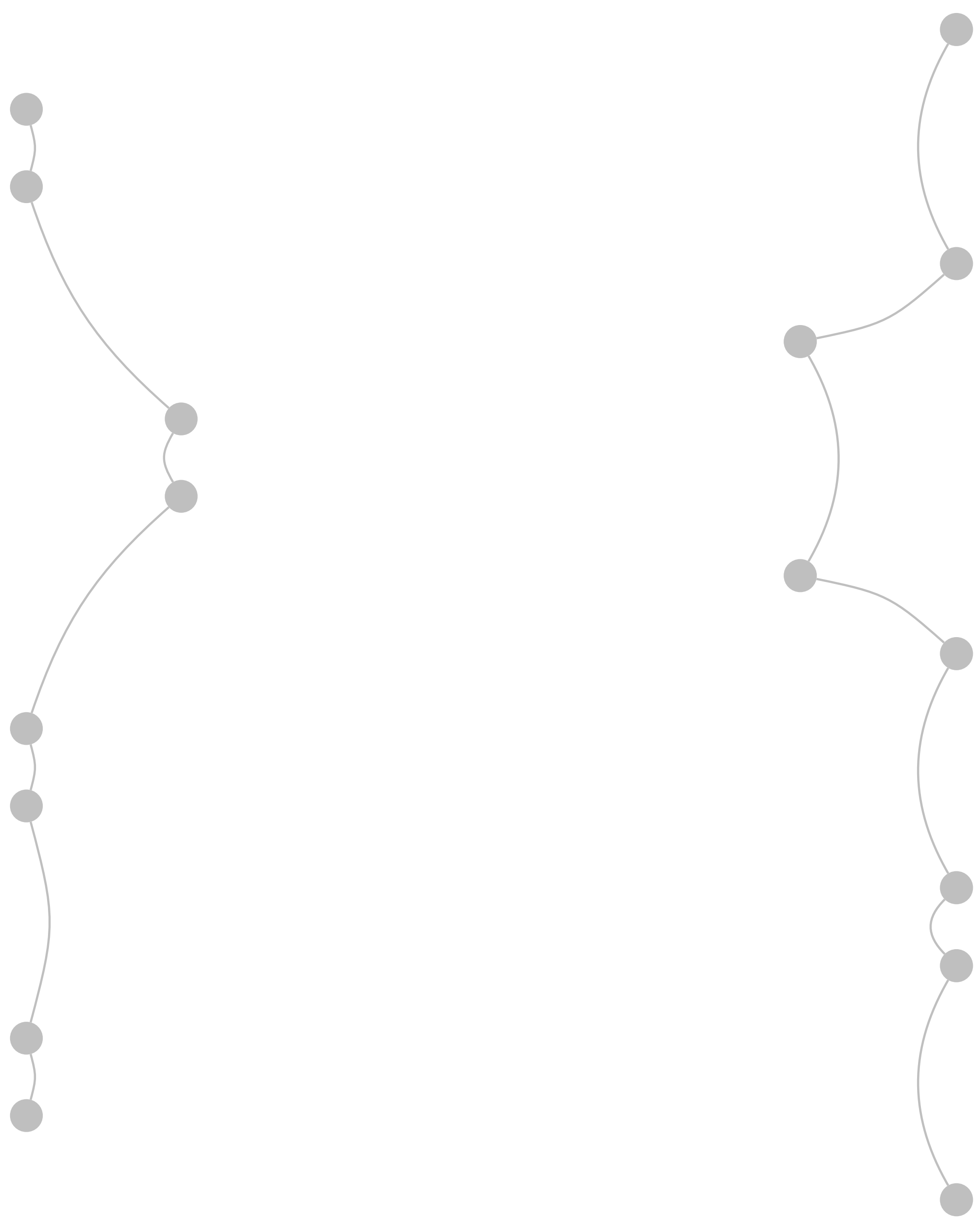
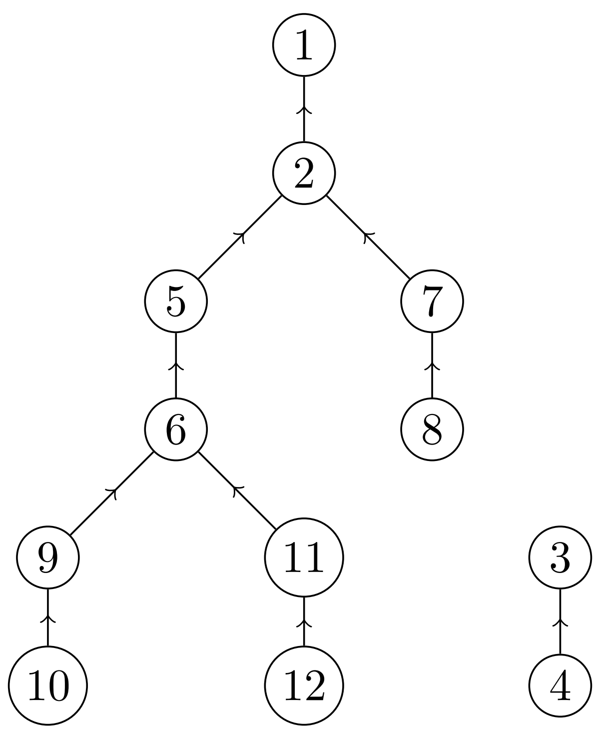


Pointers and (parity) heaps



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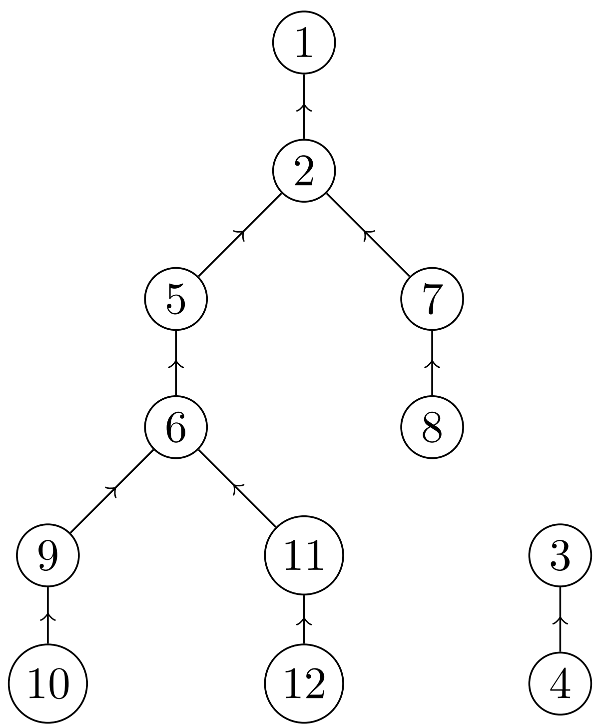

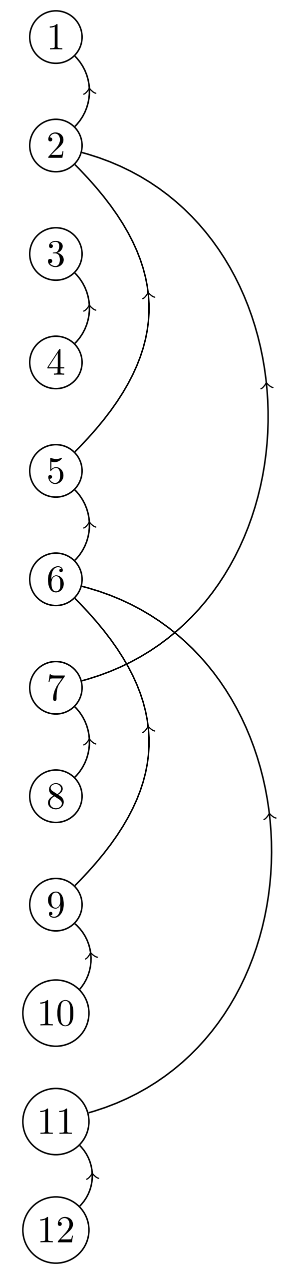

Heap graphs

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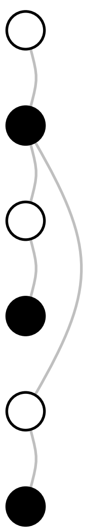

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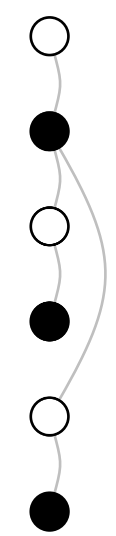

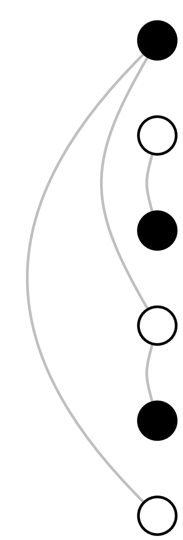

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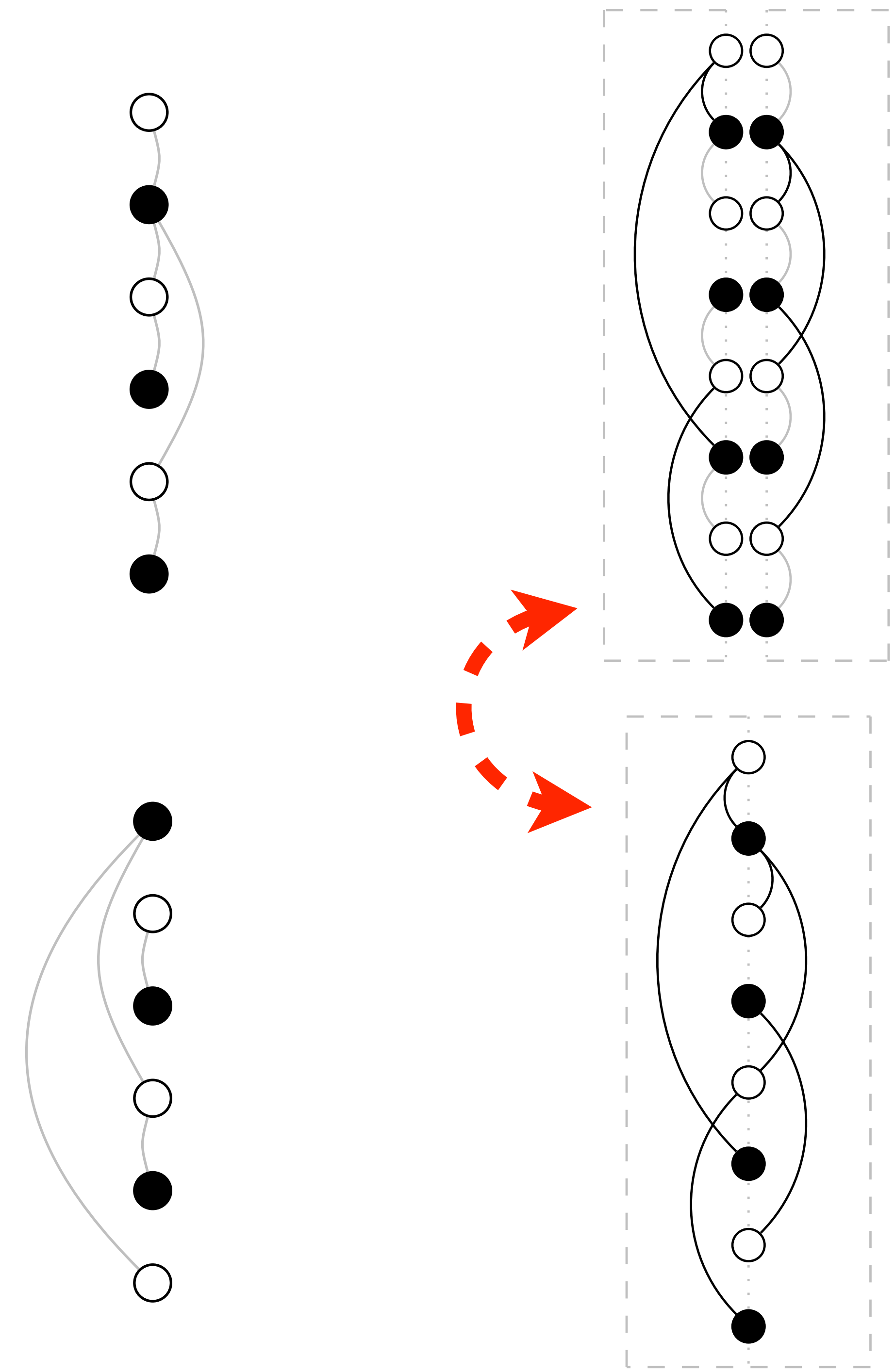
- Any parity heap graph can be composed/decomposed into an O-heap and a P-heap.



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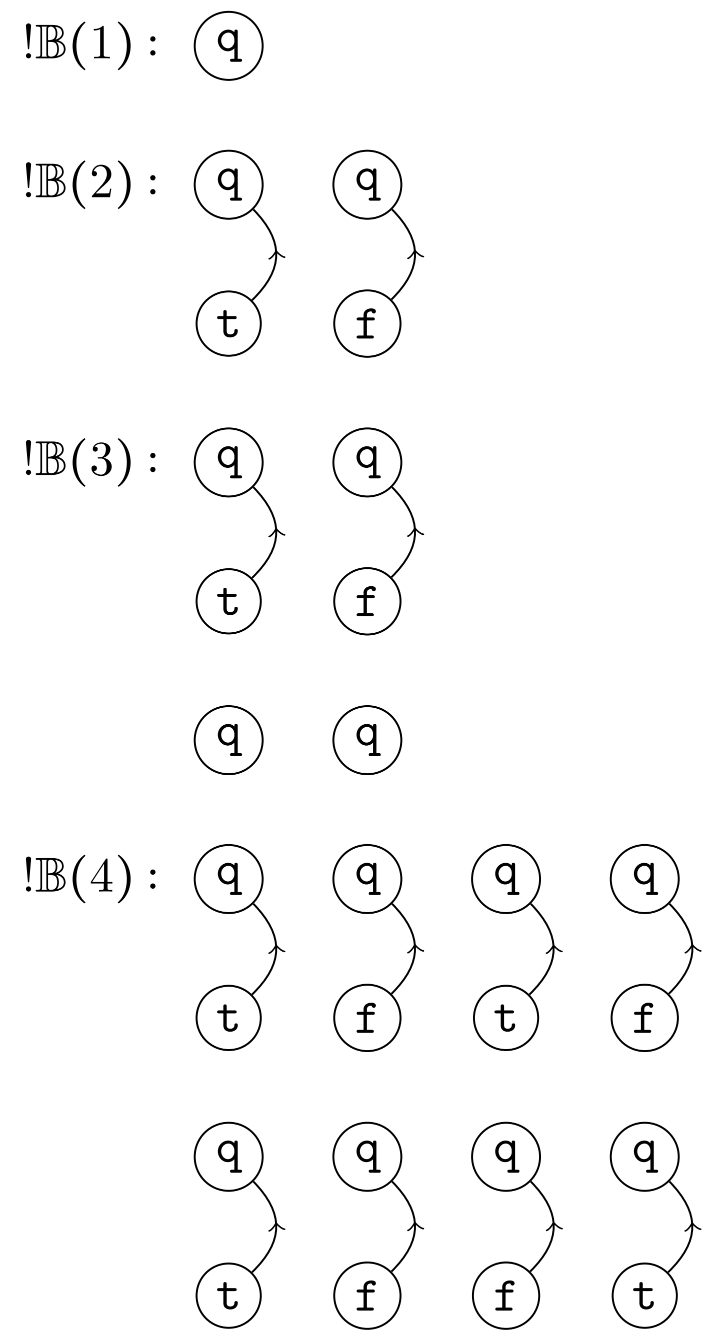

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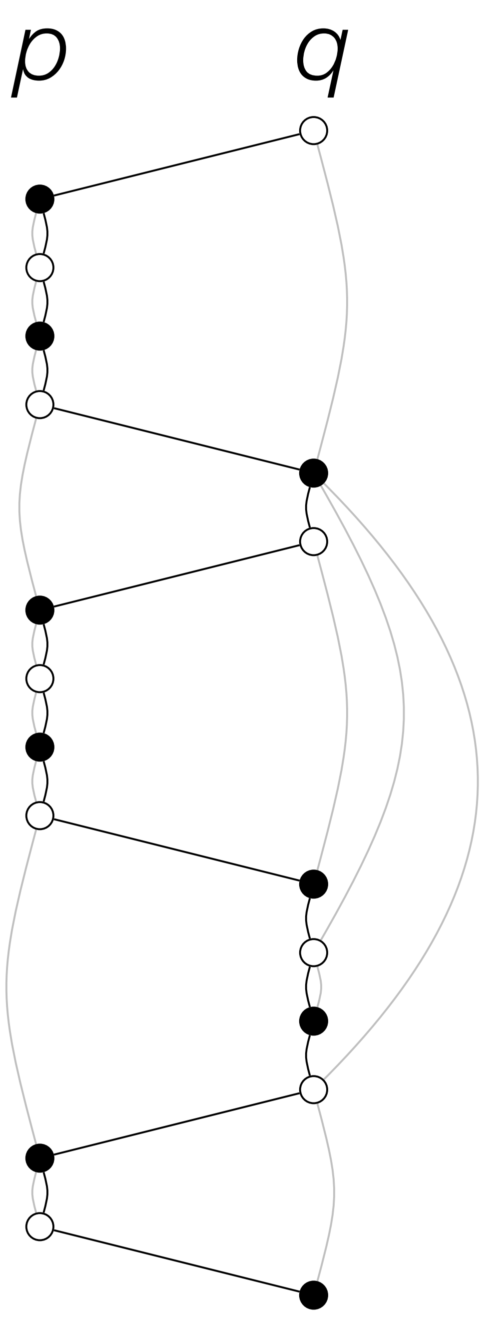

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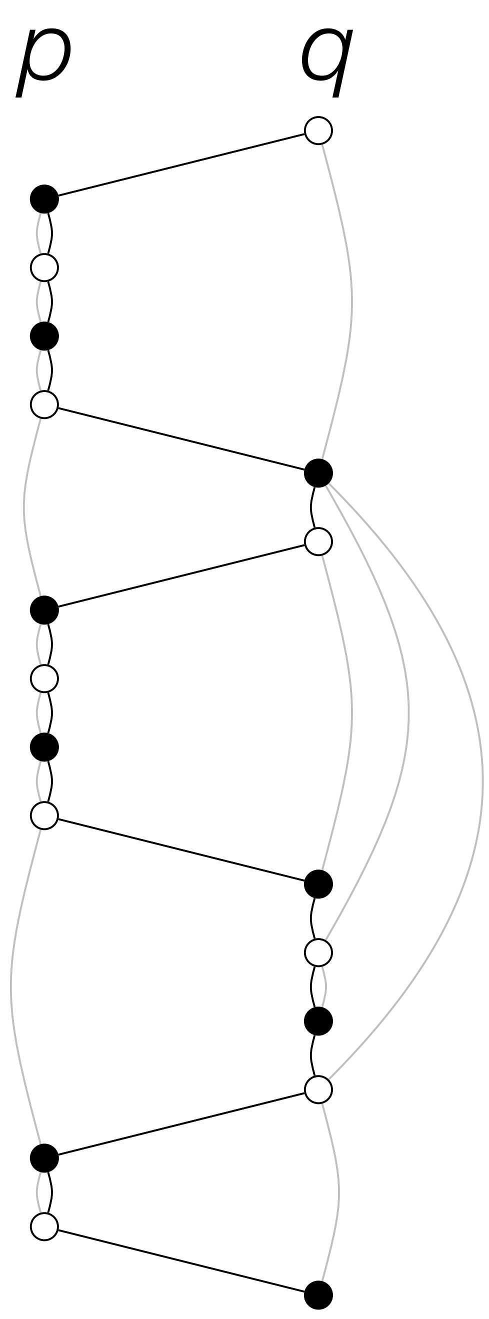

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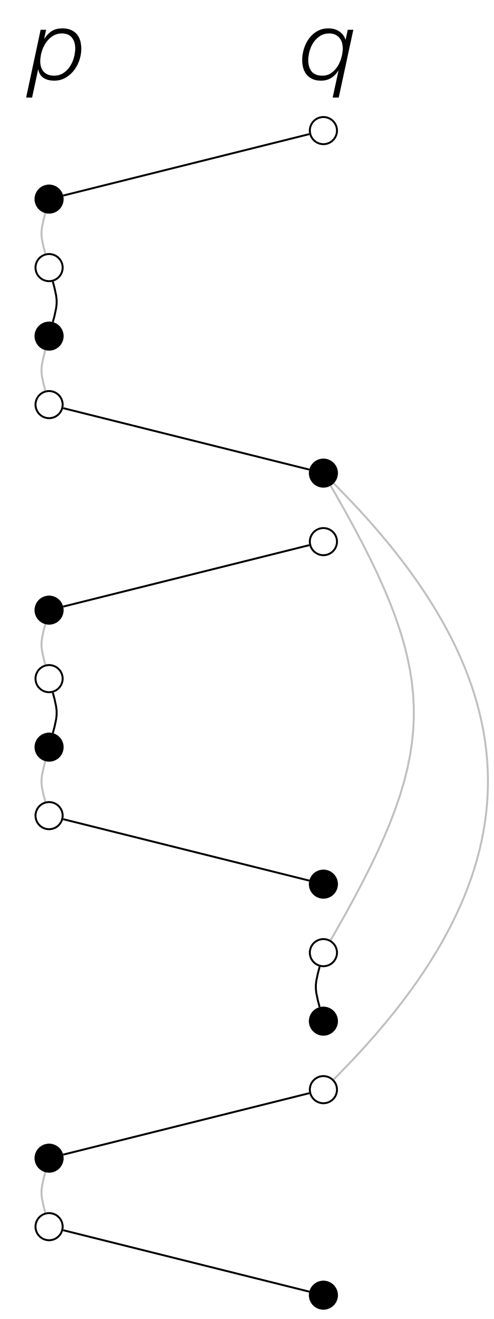

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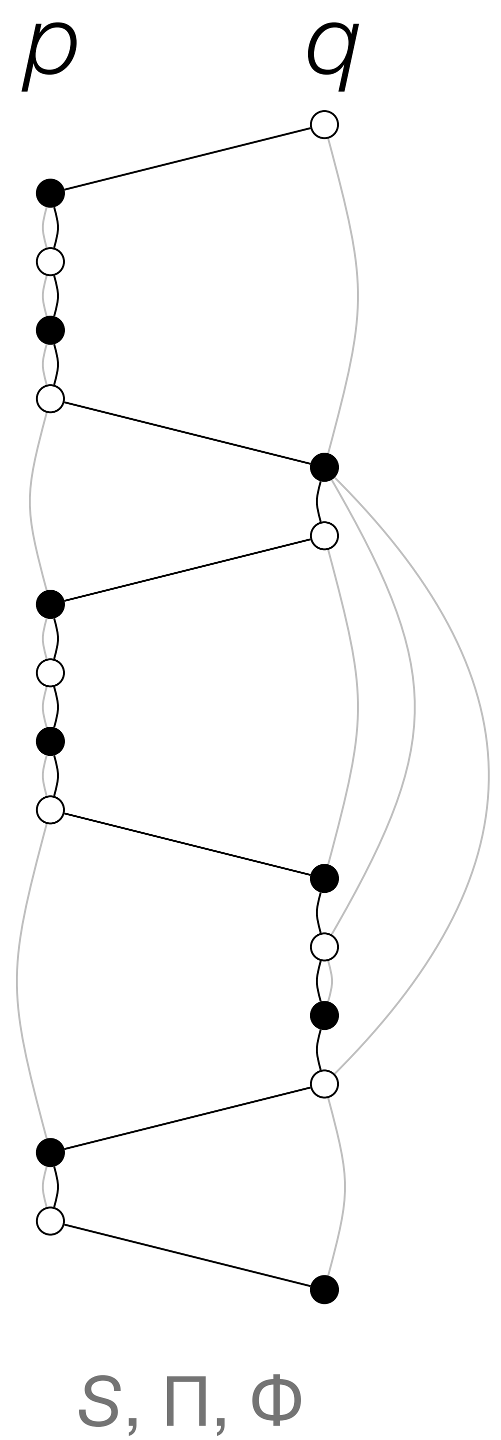

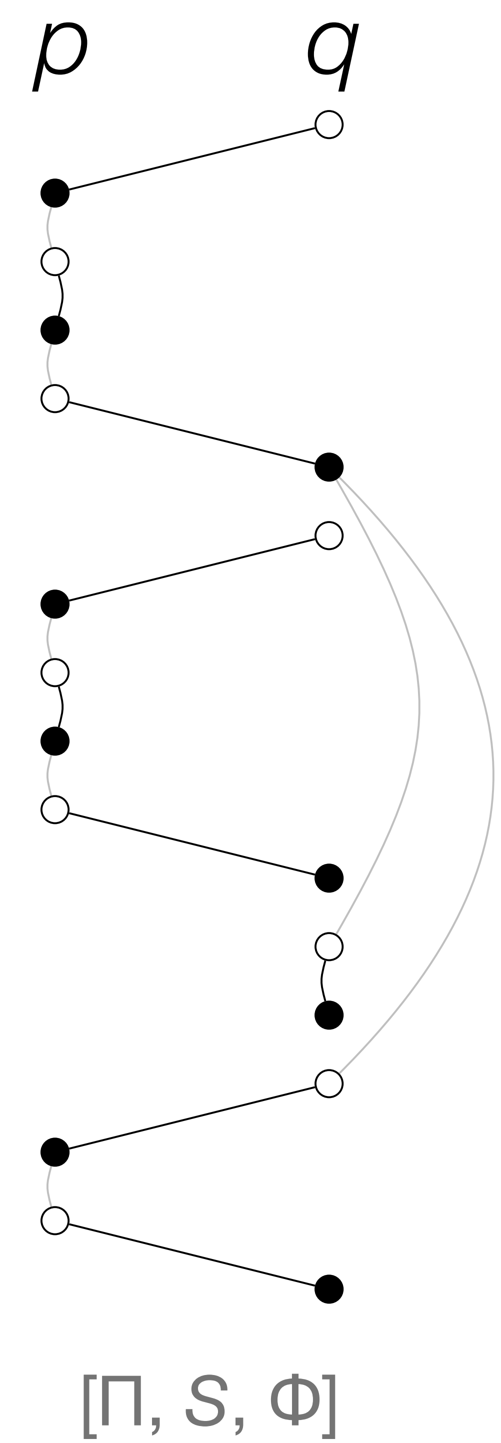

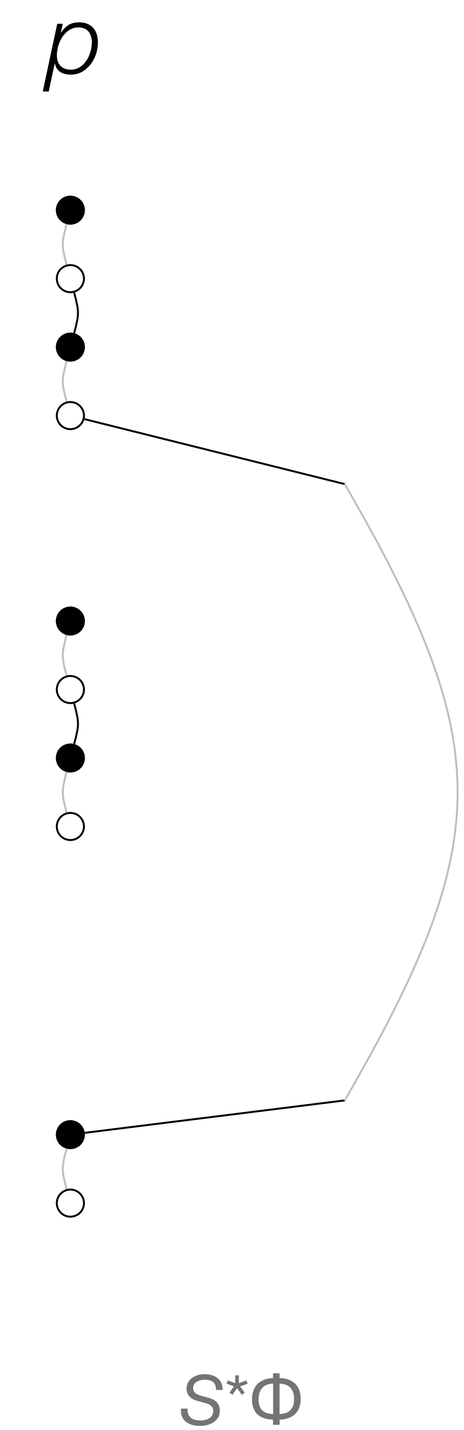

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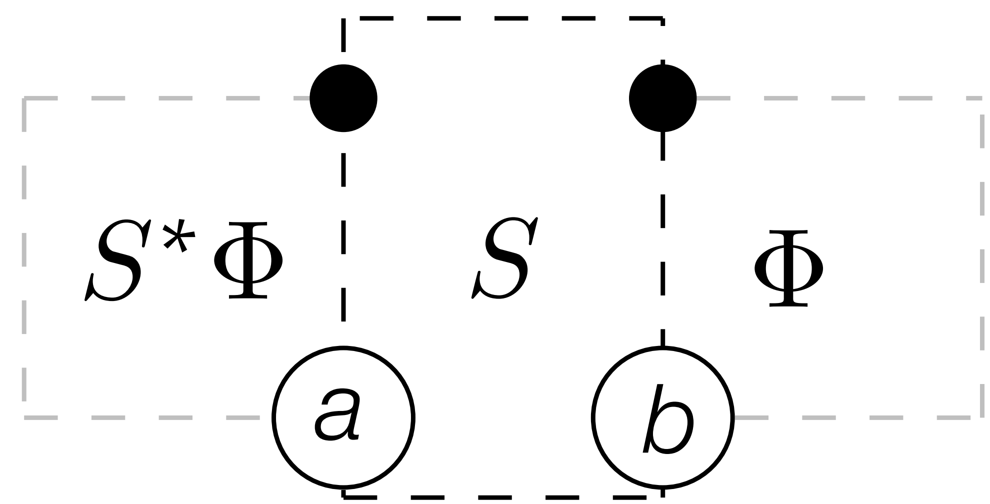

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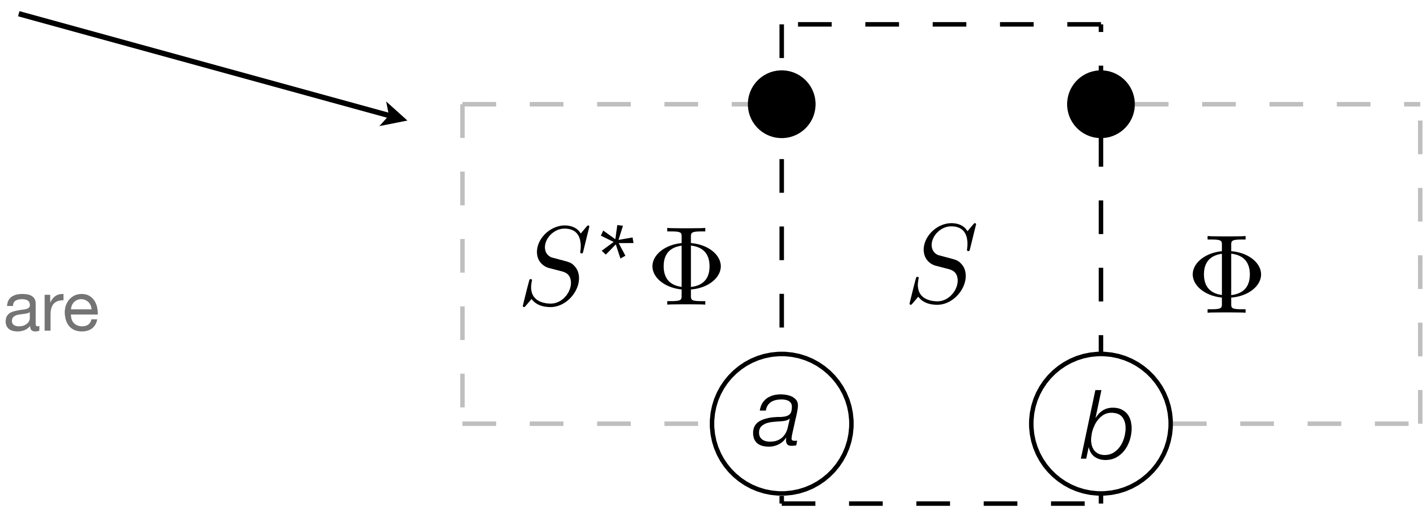

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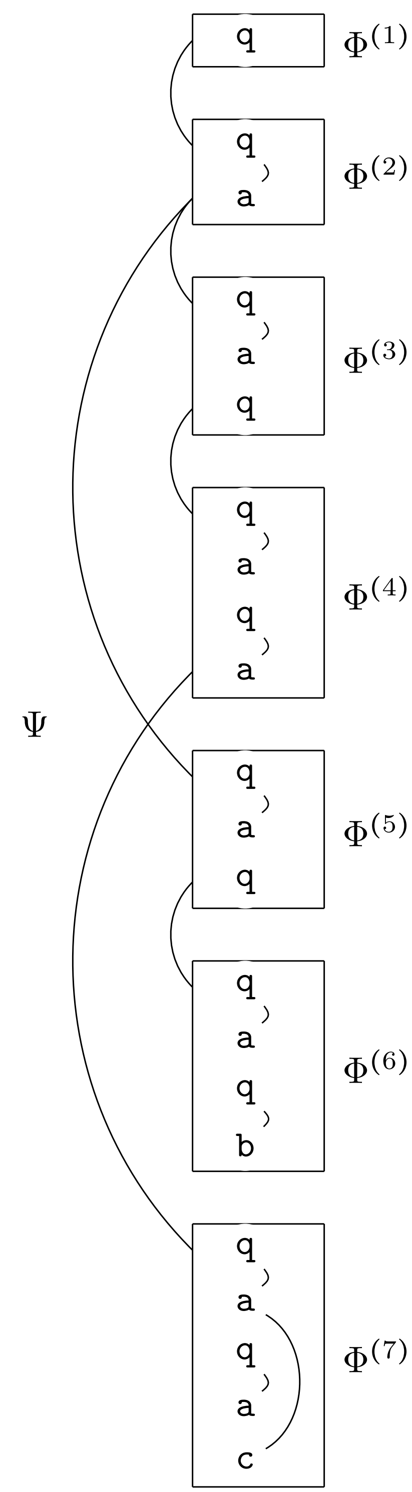

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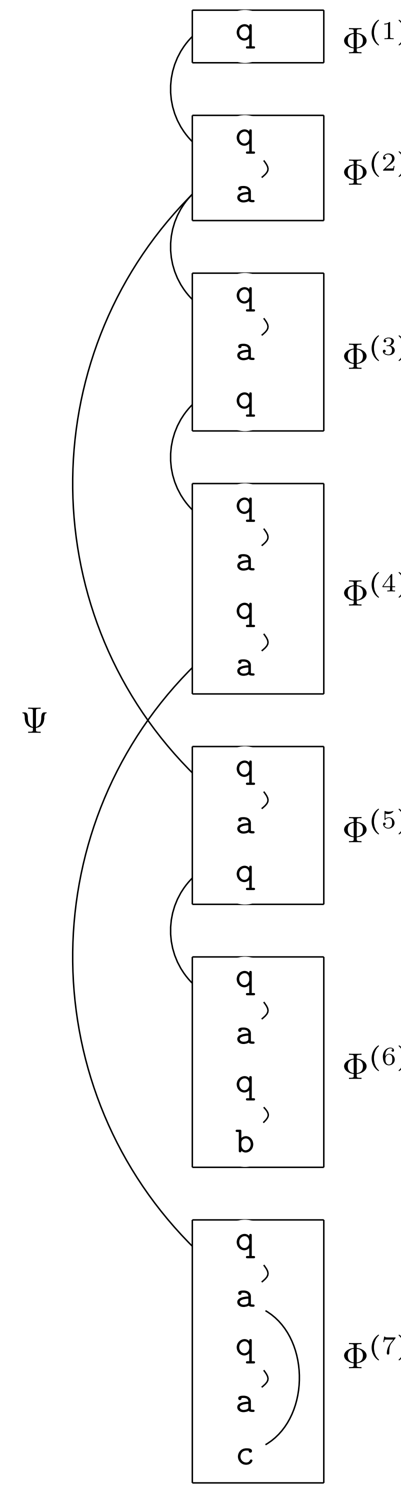

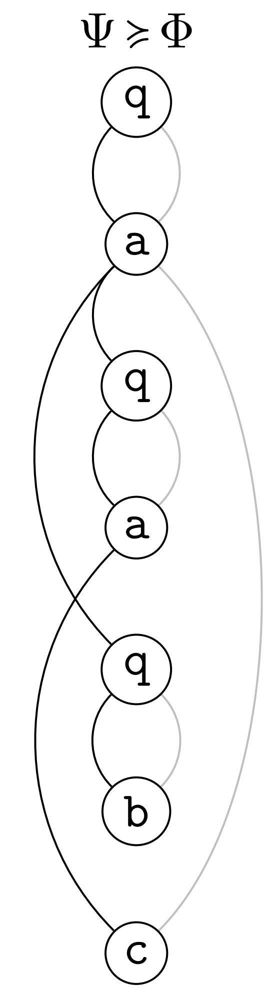

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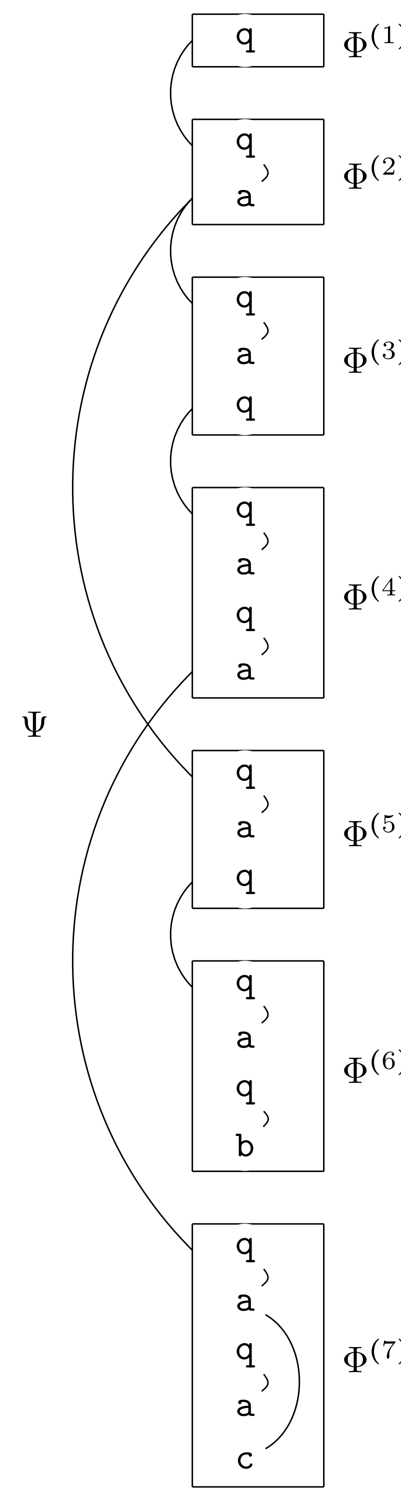

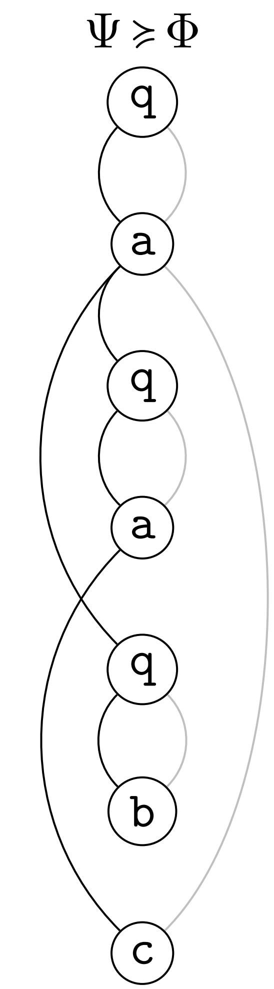

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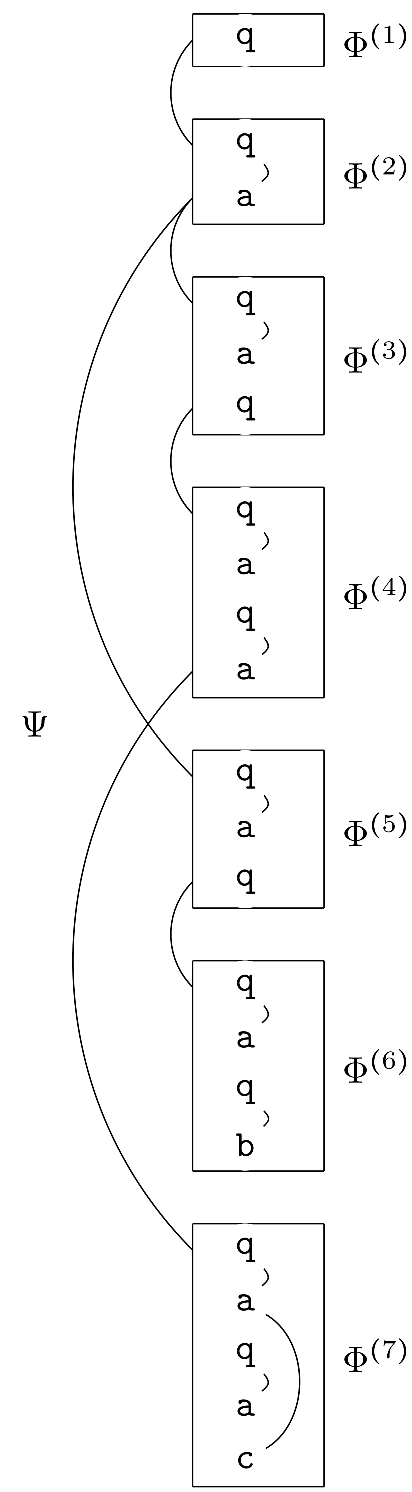

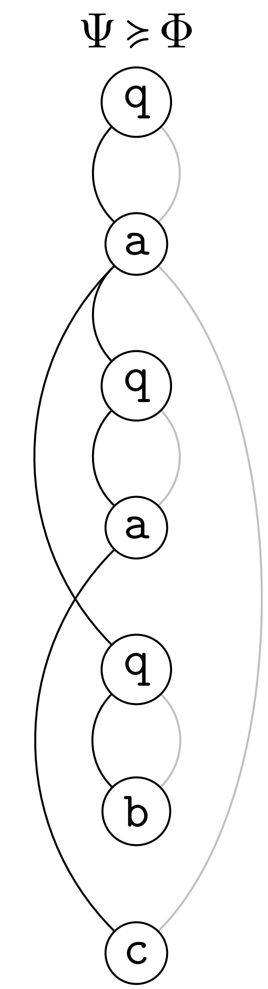

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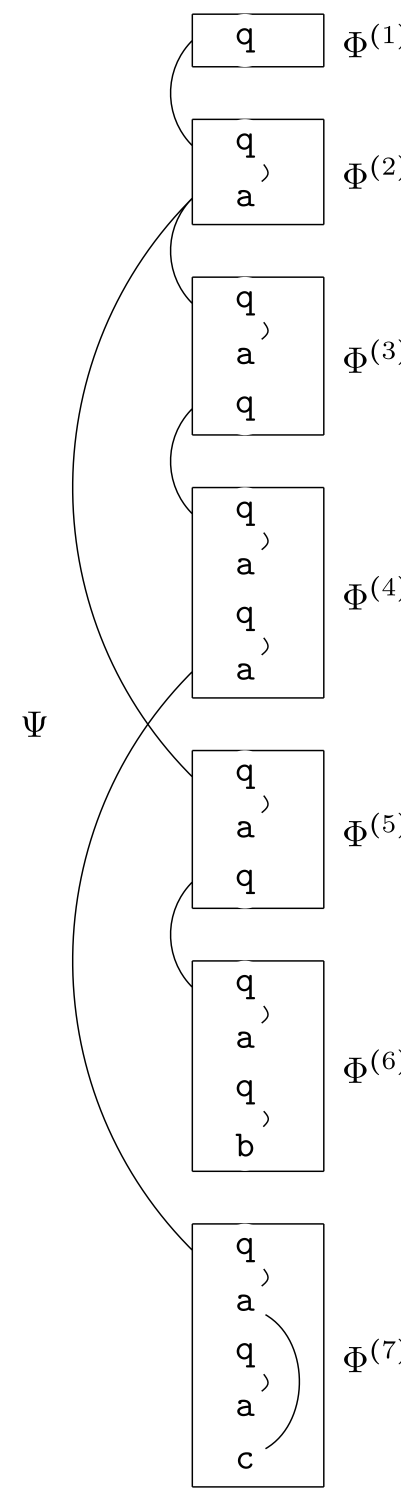
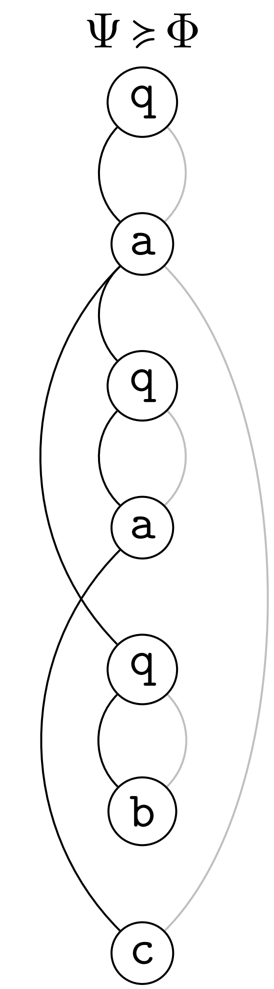
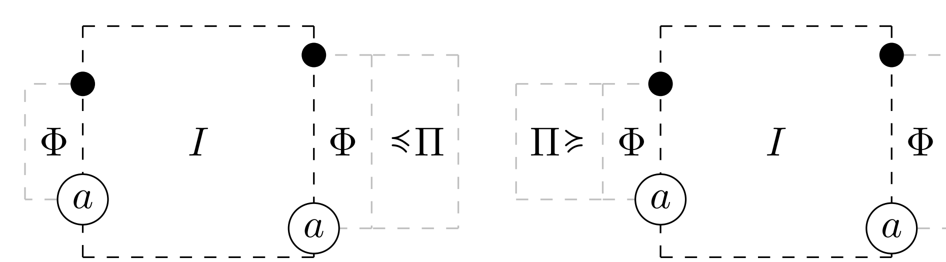




$$\delta_A \| \varepsilon_{!A} : !A \multimap !A$$



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